A note on Integer Linear Programming formulations for linear ordering problems on graphs

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Abstract

In this paper, we present a new set of constraints for modeling linear ordering problems on graphs using Integer Linear Programming (ILP). These constraints express the membership of a vertex to a prefix rather than the exact position of a vertex in the ordering. We use these constraints to propose new ILP formulations for well-known linear ordering optimization problems, namely the Pathwidth, Cutwidth, Bandwidth, SumCut and Optimal Linear Arrangement problems. Our formulations are not only more compact than previous proposals, but also more efficient as shown by our experimental evaluations on large benchmark instances.

1 Introduction

Linear ordering problems on graphs refer to optimization problems in which one aim at finding a bijection π from the set V of vertices of the graph to the set of integers $\{1, 2, \cdots, n\}$, with $n = |V|$, that minimizes a cost function $f(G, \pi)$. The bijective mapping π is called a *linear ordering* and is also known as *linear layout* or simply *layout*, *permutation*, and *linear arrangement*. Examples of such optimization problems are Bandwidth, Cutwidth and Pathwidth .

The Operations Research community has investigated multiple approaches for defining and computing a bijection $\pi : V \to \{1, \cdots, n\}$ in a mathematical program. In particular, the *Constraint Programming* (CP) and the *Boolean Satisfiability Problem* (SAT) communities have defined the `all_different` predicate of constraint satisfaction that ensures that the n integer variables $\{\nu_1, \cdots, \nu_n\}$ are assigned pairwise distinct integers from $[1, n]$, unless some constraints of the problem are violated. Hence, the `all_different` predicate models the problem of finding a valid linear ordering for the input graph, assigning each vertex a unique integer (i.e., position) $\pi(u) = \nu_u$. We refer to [3, 24, 32, 37–39] for more information on the `all_different` predicate.

In this paper, we present a new set of constraints for modeling linear ordering problems on graphs that we use to propose new Integer Linear Programming (ILP) formulations for some linear ordering optimization problems on graphs, namely the Pathwidth, Cutwidth, Bandwidth, SumCut and Optimal Linear Arrangement problems. We will show that these formulations are more efficient than previous proposals, thanks to our new set of constraints.

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Our work shall thus be useful for improving the implementation of the `all_different` predicate in various solvers. Indeed, behind the scene, CP/SAT solvers transform the `all_different` predicate applied to the set of variables $\{\nu_1, \cdots, \nu_n\}$ into a set of new (binary) variables and constraints (or clauses) to indicate whether $\nu_u < \nu_v$ or $\nu_u > \nu_v$. For instance, if we represent each integer ν_u as a bit-vector of size $l = \lceil \log n \rceil$, [37] shows how to reformulate the `all_different` predicate, applied to the n bit-vectors, using $O(n^2)$ *propositional* variables and $O(n^2 \lceil \log n \rceil)$ clauses ensuring that the n resulting bit-vectors will be pairwise distincts (and so the ν_u variables will be pairwise distincts).

This paper is organized as follows. We start in Section 3 with a survey on the sets of variables and constraints that are commonly used to express linear ordering problems. In particular, we present a formulation using variables and constraints expressing the exact position of a vertex in the ordering (Formulation 1), another based on the relative ordering of pairs of vertices (Formulation 3), and a new formulation using variables expressing the membership of a vertex to a prefix (Formulation 2). We also show the relationships between these models. Then, we use these formulations in Section 4 to present ILP formulations for the Pathwidth, Cutwidth, Bandwidth, SumCut and Optimal Linear Arrangement problems. Our formulations use generally less variables and constraints than previous proposals for the same problems. After, in Section 5, we conduct an experimental evaluation of our ILP formulations using large benchmark instances, and we compare their performances with respect to other formulations found in the literature. We will show that formulations based on the membership of a vertex to a prefix are much more efficient than other proposals. Finally, we conclude this paper in Section 6 with a discussion on possible directions for future studies.

2 Notations and definitions

All graphs considered in this paper are simple, connected and loopless. A graph $G = (V, E)$ has $n = |V|$ vertices and $m = |E|$ edges. For any set $S \subseteq V$, we let $N_G(S)$ be the set of vertices in $V \setminus S$ that have a neighbor in S. The neighborhood of a vertex $u \in V$ is denoted $N_G(u)$ and its degree $\deg_G(u) = |N_G(u)|$. The maximum degree of the graph is denoted $\Delta_G = \max_{u \in V} \deg_G(u)$. We omit the subscript when there is no ambiguity. We denote uv the edge between vertices u and v, and u and v are called the *endpoints* of edge uv. We denote $G[S]$ the subgraph of $G = (V, E)$ *induced* by $S \subseteq V$, i.e., such that $V_{G[S]} = S$ and $E_{G[S]} = \{uv \mid uv \in E \text{ and } u, v \in S\}$.

A *linear ordering* (or *layout*) of a graph $G = (V, E)$ is a bijection $\pi : V \to [1, n] = \{1, 2, \cdots, n\}$, with $n = |V|$. So, $\pi(u)$ is the position (or index) of vertex u in the linear ordering π, and $\pi^{-1}(i)$ is the vertex with position i in π. We denote $\Pi(V)$ the set of all permutations of V and so of all possible linear orderings for G. As in [5], we denote $\pi_{<,v}$ the set of vertices that appear before vertex v in the ordering, and so $\pi_{<,v} = \{u \in V \mid \pi(u) < \pi(v)\}$. We define similarly $\pi_{\leq,v}$, $\pi_{>,v}$, and $\pi_{\geq,v}$, and we write $\pi_{*,i}$ instead of $\pi_{*,\pi^{-1}(i)}$ for each operator $* \in \{<, \leq, \geq, >\}$.

We now define some measures on a linear ordering $\pi \in \Pi(V)$ of $G = (V, E)$. The cardinality of the *edge-cut* at position i in the ordering π is defined as $\theta(\pi, i) = |\{uv \in E \mid u \in \pi_{\leq,i} \wedge v \in \pi_{>i}\}|$, and the cardinality of the *modified edge-cut* at position i as $\zeta(\pi, i) = |\{uv \in E \mid u \in \pi_{<,i} \wedge v \in \pi_{>i}\}|$. The cardinality of the *vertex-cut* at position i is $|N(\pi_{\leq,i})|$. We also define the *length* of edge uv in the linear ordering π as $\lambda(\pi, uv) = |\pi(u) - \pi(v)|$.

Using above definitions, we are now ready to give the definition, as linear ordering problems, of the optimization problems on graphs that are considered in this paper. Given a graph $G = (V, E)$, each of these optimization problems asks to find a linear ordering $\pi \in \Pi(V)$

minimizing a particular cost function $f(G,\pi)$, and so the optimum value for an optimization problem is $f(G) = \min_{\pi\in\Pi(V)} f(G,\pi)$. We summarize in Table 1 the cost function of each of these optimization problems, as well as the notations for the optimal values.

Problem name	Optimum	Cost function
Pathwidth	$\mathrm{pw}(G)$	$\max_{i=1}^{n} \lvert N(\pi_{\leq,i})\rvert$
Cutwidth	$\mathrm{cw}(G)$	$\max_{i=1}^{n} \theta(\pi,i)$
Modified Cutwidth	$\mathrm{mcw}(G)$	$\max_{i=1}^{n} \zeta(\pi,i)$
Bandwidth	$\mathrm{bw}(G)$	$\max_{uv\in E} \lambda(\pi,uv)$
Optimal Linear Arrangement	$\mathrm{ola}(G)$	$\sum_{uv\in E} \lambda(\pi,uv) = \sum_{i=1}^{n} \theta(\pi,i)$
SumCut	$\mathrm{sc}(G)$	$\sum_{i=1}^{n} \lvert N(\pi_{\leq,i})\rvert$
Profile	$\mathrm{pr}(G)$	$\sum_{u\in V} \max\left\{0, \pi(u) - \min_{v\in N(u)} \pi(v)\right\}$

Table 1: Graph linear ordering optimization problems and associated cost functions.

The optimization problems presented in Table 1 are well known to be NP-complete for general graphs [12,16,17,30], and we refer to [30] for a recent state-of-the-art survey on the complexity of these problems on particular graph classes. Notice that the Profile and the SumCut problems are known to be equivalent and so we have $\mathrm{sc}(G) = \mathrm{pr}(G)$. This result follows from the fact that a vertex u belongs to $\max\left\{0, \max_{v\in N(u)} \pi(u) - \pi(v)\right\}$ sets $N(\pi_{\leq,i})$. The SumCut problem is also known to be equivalent to the Minimum Interval Graph Completion problem which asks for the minimum cardinality set of edges F such that the graph $G' = (V, E\cup F)$ is an interval graph. Moreover, the Optimal Linear Arrangement problem is also known as the Minimum Linear Arrangement problem.

3 Linear ordering constraints

In this section, we present several approaches for modeling a linear ordering problem on graph using integer linear programing. These formulations will be used in Section 4 for modeling the linear ordering optimization problems on graphs defined in Section 2 with ILPs.

A classical method for modeling a linear ordering problem (or the `all_different` predicate) using an ILP is Formulation 1 which has been used by many authors for modeling various linear ordering optimization problems on graphs. For instance, it has been used for the Bandwidth [26,31], Cutwidth [26,28], Pathwidth [20], and Optimal Linear Arrangement [1,2,29,34] problems.

Formulation 1

$$\sum_{i=1}^{n} a_u^i = 1 \qquad \forall u\in V \tag{1a}$$

$$\sum_{u\in V} a_u^i = 1 \qquad \forall i\in[1,n] \tag{1b}$$

where binary variable $a_u^i = 1$ if $\pi(u) = i$ and 0 otherwise.

Constraint (1a) ensures that vertex u is assigned a single position and Constraint (1b) ensures that position i is assigned to a single vertex. When $a_u^i = 1$, we have $\sum_{j=1}^{n} j \cdot a_u^j = i \cdot a_u^i = i$. So, the linear ordering $\pi \in \Pi(V)$ yielded by Formulation 1 is such that $\pi(u) = \sum_{i=1}^{n} i \cdot a_u^i$. Formulation 1 uses n^2 binary variables and $2n$ constraints.

The next formulation uses binary variables encoding the membership of a vertex to $\pi_{\leq,i}$.

Formulation 2

$$x_u^i \leq x_u^{i+1} \qquad \forall u \in V,\ \forall i \in [1, n-1] \tag{2a}$$

$$\sum_{u \in V} x_u^i = i \qquad \forall i \in [1, n] \tag{2b}$$

where binary variable $x_u^i = 1$ *if* $\pi(u) \leq i$ *and 0 otherwise.*

Constraints (2a) and (2b) ensure that all vertices get different positions. Indeed, Constraint (2a) ensures that as soon as $x_u^i = 1$, then $x_u^j = 1$ for all $j \in [i+1, n]$. Also, if $\pi(u) = i$, we have $\sum_{j=1}^{n} x_u^j = \sum_{j=i}^{n} x_u^j = n - i + 1$. Furthermore, Constraint (2b) ensures that i vertices are assigned a position in $[1, i]$, i.e., $|\pi_{\leq,i}| = |\{u \in V \mid \pi(u) \leq i\}| = i$. Consequently, Formulation 2 yields a linear ordering $\pi \in \Pi(V)$ such that $\pi(u) = n + 1 - \sum_{i=1}^{n} x_u^i$. Formulation 2 uses n^2 binary variables and n^2 constraints.

To the best of our knowledge, Formulation 2 has only been used in [35] (actually a variant in which Constraint (2a) is replaced with "$x_u^i \leq x_u^j$ for all $u \in V$ and $1 \leq i < j \leq n$" and so using $O(n^3)$ constraints) to propose an ILP formulation for the so-called *lightpath reconfiguration* problem in optical networks. This problem is equivalent to the Vertex Separation problem in directed graphs [8–11], and so to the Pathwidth problem in undirected graphs [22].

Another approach for obtaining a linear ordering is to use the relative ordering of pairs of vertices as follows.

Formulation 3

$$\alpha_{u,v} + \alpha_{v,u} = 1 \qquad \forall u, v \in V \tag{3a}$$

$$\alpha_{u,v} + \alpha_{v,w} - \alpha_{u,w} \leq 1 \qquad \forall u, v, w \in V \tag{3b}$$

where binary variable $\alpha_{u,v} = 1$ *if* $\pi(u) < \pi(v)$ *and 0 otherwise.*

Constraints (3a) and (3b) ensure that all vertices get different positions. Indeed, Constraint (3a) ensures that each pair of vertices is ordered, and Constraint (3b) ensures the transitivity between variables and so a total ordering, i.e., if $\pi(u) < \pi(v)$ and $\pi(v) < \pi(w)$ then $\pi(u) < \pi(w)$. Now observe that if $\pi(u) = i$, $i - 1$ vertices are positioned before u and so $\sum_{v \in V | v \neq u} \alpha_{v,u} = i - 1$. Hence, Formulation 3 yields a linear ordering $\pi \in \Pi(V)$ such that $\pi(u) = 1 + \sum_{v \in V | v \neq u} \alpha_{v,u}$. Formulation 3 uses n^2 binary variables and $O(n^3)$ constraints.

Formulation 3 has been used to propose an ILP formulation for the Treewidth problem using $O(n^3)$ binary variables and $O(n^3)$ constraints [23] (see [14] for more details on this formulation). It has also been used in [25] to obtain an ILP formulation for the Cutwidth problem using $O(n^4)$ variables and $O(n^4)$ constraints. ILP formulations for the Minimum Linear Arrangement problem using $O(n^2)$ variables and $O(n^3)$ constraints can be found in [34]. We also refer to [18, 19] and references therein for more insight on Formulation 3.

Relationships between formulations. Observe that given the assignment of the variables a_u^i provided by Formulation 1, we can deduce an assignment of the variables x_u^i of Formulation 2 using $x_u^i = \sum_{j=1}^{i} a_u^j$. Similarly, we can deduce an assignment of the variables a_u^i from the assignment of the variables x_u^i provided by Formulation 2 using $a_u^1 = x_u^1$ and $a_u^k = x_u^k - x_u^{k-1}$, with $k \in [2, n]$.

Now, given an assignment of the variables $\alpha_{u,v}$ provided by Formulation 3 we can construct an assignment of the variables x_u^i. Indeed, we have $x_u^i = 1$ if $\sum_{v\in V|v\neq u} \alpha_{v,u} < i$, i.e., if at most $i-1$ vertices are positioned before u in π. Also, introducing intermediate variables $p_u = 1+\sum_{v\in V|v\neq u} \alpha_{v,u}$ storing the position of u, we have $x_u^i = 1$ if $i+1-p_u > 0$ and 0 otherwise, and so we use the constraint $i \cdot x_u^i \geq i + 1 - p_u$, for all $u \in V$ and $i \in [1, n]$, combined with the constraint $\sum_{u\in V} \sum_{i=1}^{n} x_u^i = n(n + 1)/2$. Conversely, given an assignment of the variables x_u^i provided by Formulation 2 we can obtain an assignment of the variables $\alpha_{u,v}$. Again, we use intermediate variables $p_u = n+1-\sum_{i=1}^{n} x_u^i$ to store the position of u, and we use the constraint $n \cdot \alpha_{u,v} \geq p_v - p_u$ combined with $\sum_{v,v\in V|u\neq v} \alpha_{u,v} = n(n-1)/2$.

In Table 2, we summarize the formulations of some operations that will be useful in the next sections for modeling linear ordering optimization problems on graphs with respectively Formulations 1 to 3. We observe that some operations that are easy to model with one formulation are more complex with another formulation. For instance, it is easy to test if $\pi(u) < \pi(v)$ with Formulation 3 (it suffices to check whether variable $\alpha_{u,v} = 1$ or not) while Formulations 1 and 2 require to sum $2n$ variables. On the other hand, with Formulation 2 it suffices to check whether variable $x_u^i = 1$ or not to know if $u \in \pi_{\leq,i}$, while Formulation 1 requires to sum i variables and Formulation 3 n variables.

		Formulation 1	Formulation 2	Formulation 3
$\pi(u)$	is	$\sum_{i=1}^{n} i \cdot a_u^i$	$n+1-\sum_{i=1}^{n} x_u^i$	$\sum_{v\in V} \alpha_{v,u}$
$\pi(u) - \pi(v)$	is	$\sum_{i=1}^{n} i \cdot (a_u^i - a_v^i)$	$\sum_{i=1}^{n} (x_v^i - x_u^i)$	$\sum_{w\in V\setminus\{u\}} \alpha_{w,u} - \sum_{w\in V\setminus\{v\}} \alpha_{w,v}$
$\pi(u) < \pi(v)$	True if	$\sum_{i=1}^{n} i \cdot (a_v^i - a_u^i) > 0$	$\sum_{i=1}^{n} (x_u^i - x_v^i) > 0$	$\alpha_{u,v} = 1$
$\pi(u) = i$	True if	$a_u^i = 1$	$x_u^i - x_u^{i-1} = 1$	$\sum_{v\in V} \alpha_{v,u} = i$
$\pi(u) \leq i$	True if	$\sum_{j=1}^{i} a_u^j = 1$	$x_u^i = 1$	$\sum_{v\in V} \alpha_{v,u} \leq i$
$\pi(u) \leq i < \pi(v)$	True if	$\sum_{j=1}^{i} (a_u^j - a_v^j) = 1$	$x_u^i - x_v^i = 1$	$\begin{cases} \sum_{w\in V\setminus\{u\}} \alpha_{w,u} \leq i-1 \\ \sum_{w\in V\setminus\{v\}} \alpha_{w,v} \geq i \end{cases}$

Table 2: Useful operations when using the variables of Formulations 1 to 3.

4 ILP formulations for graph linear ordering optimization problems

In this section, we present ILP formulations for several linear ordering optimization problems on graphs, based on the formulations presented in Section 3. We summarize in Table 3 the number of variables and constraints in each of these formulations, as well as previously proposed formulations.

				Formulation 1			Formulation 2			Formulation 3		
	Ref.	#var.	#constr.		#var.	#constr.		#var.	#constr.		#var.	#constr.
	[13, 20]	$O(n^2m)$	$O(n^2m)$									
Pathwidth	[4]	$O(nm)$	$O(nm)$	PW1	$O(n^2)$	$O(nm)$	PW2	$O(n^2)$	$O(nm)$	PW3	$O(n^2)$	$O(n^3)$
	[35]	$O(n^2)$	$O(n^3)$									
Cutwidth	[26]	$O(n^2m)$	$O(n^2m)$	CW1	$O(nm)$	$O(nm)$	CW2	$O(nm)$	$O(nm)$	CW3	$O(nm)$	$O(n^3)$
	[25]	$O(n^3)$	$O(n^3)$	CW'1	$O(n^2)$	$O(n^2)$	CW'2	$O(n^2)$	$O(n^2)$			
	[26]	$O(n^2m)$	$O(n^2m)$									
Bandwidth	[4]	$O(n^2)$	$O(nm)$	BW1	$O(n^2)$	$O(n+m)$	BW2	$O(n^2)$	$O(n^2)$	BW3	$O(n^2)$	$O(n^3)$
	[31]	$O(n^2)$	$O(n+m)$									
SumCut				SC1	$O(n^2)$	$O(n+m)$	SC2	$O(n^2)$	$O(n^2)$	SC3	$O(n^2)$	$O(n^3)$
Profile				PR1	$O(n^2)$	$O(n+m)$	PR2	$O(n^2)$	$O(n^2)$	PR3	$O(n^2)$	$O(n^3)$
	[2]	$O(n^4)$	$O(n^3)$	OLA1	$O(n^2)$	$O(n+m)$	OLA2	$O(n^2)$	$O(n^2)$	OLA3	$O(n^2)$	$O(n^3)$
Opt. Lin. Arr.	[6]	$O(n^2)$	$O(n^2m)$	OLA'1	$O(nm)$	$O(nm)$	OLA'2	$O(nm)$	$O(nm)$	OLA'3	$O(nm)$	$O(n^3)$
	[1, 29]	$O(n^2)$	$O(n^3)$	OLA"1	$O(n^2)$	$O(n^2)$	OLA"2	$O(n^2)$	$O(n^2)$			

Table 3: Comparison of the number of variables and constraints of the ILP formulations.

4.1 Pathwidth

Given a graph $G = (V, E)$, the Pathwidth problem asks to find a linear ordering $\pi \in \Pi(V)$ that minimizes $\max_{i=1}^{n} |\{u \in V \mid \exists\, uv \in E \text{ such that } \pi(u) < i \leq \pi(v)\}| = \max_{i=1}^{n} |N(V \setminus \pi_{<,i})| = \max_{i=1}^{n} |N(\pi_{\leq,i})|$. The optimal value of this problem is $\mathrm{pw}(G) = \min_{\pi\in\Pi(G)} \max_{i=1}^{n} |N(\pi_{\leq,i})|$.

Several ILP formulations for the Pathwidth problem have already been proposed. The formulation presented in [13, 20] is based on Formulation 1, and it uses overall $O(n^2m)$ variables and constraints. Indeed, it uses n^2 binary variables per edge $uv \in E$ (i.e., $y_{uv}^{k,l} = 1$ if $a_u^k = 1$ and $a_v^l = 1$), and n^2 additional binary variables are introduced to identify the vertices in $N(\pi_{\leq,i})$. The formulation proposed in [4] is based on the representation of a graph as an interval graph. Also, it associates to each vertex an interval and every edge forces the intervals of its two endpoints to intersect. The objective is to minimize the maximum number of intervals that intersect at any given point. This formulation uses $O(nm)$ variables and constraints. The formulation proposed in [35] is based on the variant of Formulation 2 we have discussed in Section 2, and uses $O(n^2)$ variables and $O(n^3)$ constraints. It is similar to Formulation 4.

To model the Pathwidth problem as an ILP, we need a set of constraints to measure $|N(\pi_{\leq,i})|$. To do so, we introduce variable $y_u^i = 1$ if $u \in \pi_{>,i}$ and at least one neighbor v of u is in $\pi_{\leq,i}$, and 0 otherwise. In other words, $y_u^i = 1$ if $u \in N(\pi_{\leq,i})$. We can thus write the following ILP formulation.

Formulation 4 Pathwidth*: Minimize p subject to either Constraints* (1a) *and* (1b) *or Constraints* (2a) *and* (2b) *and*

$$\tau(u, v, i) \leq y_u^i \qquad \forall u \in V,\ \forall v \in N(u),\ \forall i \in [1, n] \tag{4a}$$

$$\sum_{u\in V} y_u^i \leq p \qquad \forall i \in [1, n] \tag{4b}$$

where $y_u^i = 1$ if $u \in N(\pi_{\leq,i})$, and 0 otherwise, and $\tau(u, v, i)$ is either $\sum_{j=1}^{i}(a_v^j - a_u^j)$ or $x_v^i - x_u^i$ when using respectively the variables of Formulations 1 and 2.

With Constraint (4a) we identify the vertices in $\pi_{>,i}$ that are also in $N(\pi_{\leq,i})$. Indeed, if $u \in \pi_{>,i}$, then $\pi(u) > i$ and so $\sum_{j=1}^{i} a_u^j = 0$ (resp. $x_u^i = 0$). If furthermore $v \in N(u)$ is such that $v \in \pi_{\leq,i}$, then $\sum_{j=1}^{i} a_v^j = 1$ (resp. $x_v^i = 1$) and we have $y_u^i \geq \sum_{j=1}^{i} a_v^j - \sum_{j=1}^{i} a_u^j = 1$ (resp. $y_u^i \geq x_v^i - x_u^i = 1$). With Constraint (4b) we count the number of vertices in $N(\pi_{\leq,i})$. The optimal solution of Formulation 4 is $\mathrm{pw}(G)$. This formulation adds $O(n^2)$ variables and $O(nm)$ constraints to the variables and constraints of Formulations 1 and 2.

In the following, we refer to the ILP formulations resulting from Formulation 4 when using the variables and constraints of respectively Formulations 1 and 2 as PW1 and PW2. The sizes of these ILP formulations are reported in Table 3.

Now, when using Formulation 3, we need to introduce $2n^2$ additional binary variables encoding the membership of a vertex to $\pi_{\leq,i}$ and to $\pi_{>,i}$.

Formulation 5 PATHWIDTH*: Minimize p subject to Constraints* (3a)*,* (3b) *and* (4b) *and*

$$i - \sum_{w \in V \mid w \neq u} \alpha_{w,u} \leq n \cdot l_u^i \qquad \forall u \in V,\ \forall i \in [1,n] \tag{5a}$$

$$\sum_{w \in V \setminus \{u\}} \alpha_{w,u} - i + 1 < n \cdot r_u^i \qquad \forall u \in V,\ \forall i \in [1,n] \tag{5b}$$

$$l_v^i + r_u^i - 1 \leq y_u^i \qquad \forall u \in V,\ \forall v \in N(u),\ \forall i \in [1,n] \tag{5c}$$

where binary variable $\alpha_{w,u} = 1$ if vertex $\pi(w) < \pi(u)$ and 0 otherwise, binary variable $l_u^i = 1$ if $u \in \pi_{\leq,i}$, binary variable $r_u^i = 1$ if $u \in \pi_{>,i}$, and $y_u^i = 1$ if $u \in N(\pi_{\leq,i})$, and 0 otherwise.

With the *big-M* Constraint (5a) we identify the vertices in $\pi_{\leq,i}$, with the *big-M* Constraint (5b) we identify the vertices in $\pi_{>,i}$, and with Constraint (5c) we identify the vertices in $\pi_{>,i}$ that are also in $N(\pi_{\leq,i})$. Indeed, if $u \in \pi_{>,i}$, then $\sum_{w \in V \setminus \{u\}} \alpha_{w,u} - i > 0$ and so $r_u^i = 1$. If furthermore $v \in N(u)$ is such that $v \in \pi_{\leq,i}$, then $l_v^i + r_u^i = 2$ and we have $y_u^i \geq l_v^i + r_u^i - 1 = 1$. The optimal solution of Formulation 5 is $\mathrm{pw}(G)$. This formulation uses $O(n^2)$ binary variables and $n^3 + 2n^2 + 2nm + 1 = O(n^3)$ constraints. In the following, we refer to Formulation 5 as PW3.

4.2 Cutwidth

Given a graph $G = (V, E)$, the CUTWIDTH problem asks to find a linear ordering $\pi \in \Pi(V)$ that minimizes $\max_{i=1}^{n} \theta(\pi, i) = \max_{i=1}^{n} |\{uv \in E \mid \pi(u) \leq i < \pi(v)\}|$, and so the optimal value is $\mathrm{cw}(G) = \min_{\pi \in \Pi(G)} \max_{i=1}^{n} \theta(\pi, i)$.

An ILP formulation for the CUTWIDTH problem has been proposed in [26]. This formulation, based on Formulation 1, uses n^2 binary variables per edge $uv \in E$ (i.e., $y_{uv}^{k,l} = 1$ if $a_u^k = 1$ and $a_v^l = 1$), and overall it uses $O(n^2 m)$ variables and $O(n^2 m)$ constraints. Another ILP formulation for the CUTWIDTH problem has been proposed in [25]. This formulation is based on Formulation 3. It improves upon the formulation of [26] since it uses $O(n^3)$ variables and $O(n^3)$ constraints. We will now present ILP formulations using fewer variables and constraints.

To model the CUTWIDTH problem as an ILP, we need a set of constraints to measure $\theta(\pi, i)$. To do so, a first approach is to introduce variable $z_{uv}^i = 1$ if either $\pi(u) \leq i < \pi(v)$ or $\pi(v) \leq i < \pi(u)$, and 0 otherwise. In other words, $z_{uv}^i = 1$ if one endpoint of edge uv is in $\pi_{\leq,i}$ and the other endpoint is in $\pi_{>,i}$. We can thus write the following ILP formulation.

Formulation 6 Cutwidth*: Minimize c subject to either Constraints* (1a) *and* (1b) *or Constraints* (2a) *and* (2b) *and:*

$$\tau(u,v,i) \le z^i_{uv} \qquad \forall uv \in E,\ \forall i \in [1,n] \tag{6a}$$
$$\tau(v,u,i) \le z^i_{uv} \qquad \forall uv \in E,\ \forall i \in [1,n] \tag{6b}$$
$$\sum_{uv\in E} z^i_{uv} \le c \qquad \forall i \in [1,n] \tag{6c}$$

where $z^i_{uv} = 1$ *if either* $\pi(u) \le i < \pi(v)$ *or* $\pi(v) \le i < \pi(u)$*, and 0 otherwise, and* $\tau(u,v,i)$ *is either* $\sum_{j=1}^{i}(a^j_v - a^j_u)$ *or* $x^i_v - x^i_u$ *when using respectively the variables of Formulations 1 and 2.*

Constraints (6a) and (6b) force variable z^i_{uv} to 1 if one endpoint of edge uv is in $\pi_{\le,i}$ and the other endpoint is in $\pi_{>,i}$. With Constraint (6c) we count the number $\theta(\pi,i)$ of edges in the cut $(\pi_{\le,i}, \pi_{>,i})$. The objective is to minimize the maximum number c of edges in a cut. The optimal solution of Formulation 6 is $\mathrm{cw}(G)$. This formulation adds $O(nm)$ variables and $O(nm)$ constraints to the variables and constraints of Formulations 1 and 2. In the following, we refer to the ILP formulations resulting from Formulation 6 when using the variables and constraints of respectively Formulations 1 and 2 as CW1 and CW2. The sizes of these ILP formulations are reported in Table 3.

When using Formulation 3, as in Section 4.1 for Formulation 5, we need to introduce $2n^2$ additional binary variables l^i_u and r^i_u encoding the membership of a vertex to $\pi_{\le,i}$ and to $\pi_{>,i}$. Also, we get the following ILP formulation.

Formulation 7 Cutwidth*: Minimize c subject to Constraints* (3a)*,* (3b)*,* (5a)*,* (5b) *and* (6c) *and:*

$$l^i_u + r^i_v - 1 \le z^i_{uv} \qquad \forall uv \in E,\ \forall i \in [1,n] \tag{7a}$$
$$l^i_v + r^i_u - 1 \le z^i_{uv} \qquad \forall uv \in E,\ \forall i \in [1,n] \tag{7b}$$

where $\alpha_{w,u} = 1$ *if vertex* $\pi(w) < \pi(u)$ *and 0 otherwise,* $l^i_u = 1$ *if* $u \in \pi_{\le,i}$*,* $r^i_u = 1$ *if* $u \in \pi_{>,i}$*, and* $z^i_{uv} = 1$ *if either* $\pi(u) \le i < \pi(v)$ *or* $\pi(v) \le i < \pi(u)$*, and 0 otherwise.*

With the *big-M* Constraints (5a) and (5b) we identify the vertices in $\pi_{\le,i}$ and $\pi_{>,i}$, and with Constraints (7a) and (7b) we identify edges in the cut $(\pi_{\le,i}, \pi_{>,i})$. With Constraint (6c) we count the number $\theta(\pi,i)$ of edges in the cut $(\pi_{\le,i}, \pi_{>,i})$. The objective is to minimize the maximum number c of edges in a cut. Formulation 7 uses $O(nm)$ binary variables and $O(n^3)$ constraints. In the following, we refer to Formulation 7 as CW3. This formulation is similar to the formulation proposed in [25] but uses slightly less variables and constraints.

Another approach for measuring $\theta(\pi,i)$ is based on the observation that the difference between the edge-cuts $(\pi_{\le,i-1}, \pi_{>,i-1})$ and $(\pi_{\le,i}, \pi_{>,i})$ is the removal of the edges incident to vertex $u = \pi^{-1}(i)$ and with one endpoint in $\pi_{\le,i-1}$ (i.e., the edge-cut $(\pi_{\le,i-1}, \{u\})$), and the addition of the edges incident to vertex u and with one endpoint in $\pi_{>,i}$ (i.e., the edge-cut $(\{u\}, \pi_{>,i})$). Also, we have $\theta(\pi,i) = \theta(\pi,i-1) - |(\pi_{\le,i-1}, \{u\})| + |(\{u\}, \pi_{>,i})| = \theta(\pi,i-1) + \deg(u) - 2\cdot|(\pi_{\le,i-1}, \{u\})|$, since $\deg(u) = |(\pi_{\le,i-1}, \{u\})| + |(\{u\}, \pi_{>,i})|$. Using this observation, we can model the Cutwidth problem as the following ILP.

Formulation 8 CUTWIDTH*: Minimize c subject to either Constraints* (1a) *and* (1b) *or Constraints* (2a) *and* (2b) *and:*

$$\deg(u) \cdot \omega(u,i) - 2 \sum_{v \in N(u)} \eta(v,i) - 2 \cdot \Delta \cdot (1 - \omega(u,i)) \leq \mu^i \qquad \forall u \in V,\ \forall i \in [2,n] \tag{8a}$$

$$\sum_{u \in V} \deg(u) \cdot \omega(u,1) = c^1 \tag{8b}$$

$$c^{i-1} + \mu^i = c^i \qquad \forall i \in [2,n] \tag{8c}$$

$$c^N = 0 \tag{8d}$$

$$c^i \leq c \qquad \forall i \in [1,n] \tag{8e}$$

where variable c^i is a non-negative integer used to measure the cardinality $\theta(\pi,i)$ of the edge-cut $(\pi_{\leq,i}, \pi_{>,i})$, integer variable μ^i measures the difference $\theta(\pi,i)-\theta(\pi,i-1)$, and either $\omega(u,i) = a_u^i$ and $\eta(v,i) = \sum_{j=1}^{i-1} a_v^j$ when using Formulation 1, or $\omega(u,i) = x_u^i - x_u^{i-1}$ and $\eta(v,i) = x_v^{i-1}$ when using Formulation 2. In other words, $\omega(u,i) = 1$ when $\pi(u) = i$, and $\eta(v,i) = 1$ when $v \in \pi_{<,i}$.

We have seen that $\theta(\pi,i) = \theta(\pi,i-1) + \deg(u) - 2 \cdot |(\pi_{\leq,i-1}, \{u\})|$, when $u = \pi^{-1}(i)$. With Constraint (8a) we measure the difference in number of edges with previous position. More precisely, if vertex u is at position i, edges incident to v such that the other endpoint $v \in N(u)$ has position $j < i$ ends at position i and should no longer be taken into account. We have $\sum_{v \in N(u)} \eta(v,i)$ such edges that we subtract from previous counting. In addition, $\deg(u) - \sum_{v \in N(u)} \eta(v,i)$ edges incident to u have an endpoint in $\pi_{>,i}$, and so we must add them to the previous counting. Last, we subtract $2 \cdot \Delta \cdot (1 - \omega(u,i))$ to ensure that if $\pi(u) \neq i$, the evaluation of the left-hand side of Constraint (8a) will have no impact on the value of μ^i. Then, with Constraints (8b) to (8d) we measure the cardinality $\theta(\pi,i)$ of the edge-cuts. The objective is to minimize the maximum number c of edges in an edge-cut. Formulation 8 uses $O(n^2)$ variables and constraints. In the following, we refer to the ILP formulations resulting from Formulation 8 when using the variables and constraints of respectively Formulations 1 and 2 as CW'1 and CW'2.

From above formulations for the CUTWIDTH problem, one can easily model the MODIFIED CUTWIDTH problem which asks for a linear ordering $\pi \in \Pi(V)$ minimizing $\max_{i=1}^n \zeta(\pi,i) = \max_{i=1}^n |\{uv \in E \mid \pi(u) < i < \pi(v)\}|$. For instance, with Formulation 6, it suffices to use either $\tau'(u,v,i) = \sum_{j=1}^{i-1} a_v^j - \sum_{j=1}^{i} a_u^j = \tau(u,v,i) - a_v^i$ or $\tau'(u,v,i) = x_v^{i-1} - x_u^i$ when using respectively the variables of Formulations 1 and 2. Moreover, with Formulation 7, it suffices to use $l_u^{i-1} + r_v^i - 1 \leq z_{uv}^i$ and $l_v^{i-1} + r_u^i - 1 \leq z_{uv}^i$ instead of Constraints (7a) and (7b).

4.3 Bandwidth

Given a graph $G = (V,E)$, the BANDWIDTH problem asks to find a linear ordering $\pi \in \Pi(V)$ that minimizes the maximum distance between the endpoints of the edges of G in the linear ordering, i.e., $\lambda(\pi, uv) = |\pi(u) - \pi(v)|$, and so the optimal value is $\mathrm{bw}(G) = \min_{\pi \in \Pi(V)} \max_{uv \in E} |\pi(u) - \pi(v)|$. A direct ILP formulation for this problem is as follows.

Formulation 9 BANDWIDTH*: Minimize b subject to either Constraints* (1a) *and* (1b)*, or Constraints* (2a) *and* (2b)*, or Constraints* (3a) *and* (3b) *and*

$$\ell(u,v) \leq b \qquad \forall uv \in E \tag{9a}$$

$$\ell(v,u) \leq b \qquad \forall uv \in E \tag{9b}$$

where $\ell(u,v)$ is either $\sum_{i=1}^{n} i \cdot \left(a_u^i - a_v^i\right)$, or $\sum_{i=1}^{n} \left(x_v^i - x_u^i\right)$, or $\sum_{w\in V\setminus\{u\}} \alpha_{w,u} - \sum_{w\in V\setminus\{v\}} \alpha_{w,v}$ when using respectively the variables of Formulations 1, 2 and 3.

Constraints (9a) and (9b) ensure that the distance between the endpoints of any edge $uv \in E$ is at most b and the objective is to minimize b. The optimal solution of Formulation 9 is $\mathrm{bw}(G)$. This formulation adds $O(m)$ constraints to the constraints of Formulations 1 to 3. In the following, we refer to the ILP formulations resulting from Formulation 9 when using the variables and constraints of respectively Formulations 1 to 3 as BW1, BW2, and BW3. The sizes of these ILP formulations are reported in Table 3.

Formulation BW1 is almost the same as the formulation based on Formulation 1 proposed in [31]. The difference is that [31] introduce integer variables $p_u = \sum_{i=1}^{n} i \cdot a_u^i$, recording the position of vertex u in the linear ordering, and rewrite Constraints (9a) and (9b) as $p(u) - p(v) \leq b$ and $p(v) - p(u) \leq b$. Another ILP formulation for the Bandwidth problem based on Formulation 1 has been proposed in [26]. It uses variable $y_{uv}^{k,l} = 1$ when $a_u^k = 1$ and $a_v^l = 1$. The goal is to minimize b where $b \geq |k-l| \cdot y_{uv}^{k,l}$ for all $uv \in E$ and $k, l \in [1, n]$. Overall it uses $O(n^2 m)$ variables and constraints. A different approach has been followed in [4], proposing an ILP formulation based on the representation of a graph as an interval graph. It associates to each vertex an interval and every edge forces the intervals of its two endpoints to intersect. In this formulation, the length of an interval is bounded by a parameter k, and the objective is to find the minimum k for which a feasible solution exists. Hence, this formulation must be solved for $\log n$ values of k (or $\mathrm{bw}(G)$ times if using only growing values of k). This formulation uses $O(n^2)$ variables and $O(nm)$ constraints.

4.4 Profile and Sum Cut

We have seen in Section 2 that the SumCut, Profile and Minimum Interval Graph Completion problems are equivalent [12]. Also, in this section we present two main formulations.

First, we obtain ILP formulations for the SumCut problem from the ILP formulations for the Pathwidth problem. Indeed, the SumCut problem asks for a linear ordering $\pi \in \Pi(V)$ minimizing $\sum_{i=1}^{n} |N(\pi_{\leq,i})|$, while the Pathwidth problem asks for a linear ordering $\pi \in \Pi(V)$ minimizing $\max_{i=1}^{n} |N(\pi_{\leq,i})|$. Also, it suffices to replace, in Formulations 4 and 5, Constraint (4b) with $\sum_{i=1}^{n} \sum_{u\in V} y_u^i \leq p$. In the following, we refer to the resulting ILP formulations when using the variables and constraints of respectively Formulations 1 to 3 as SC1, SC2, and SC3. The sizes of these ILP formulations are reported in Table 3.

Next, we use the definition of the Profile problem which asks for a linear ordering $\pi \in \Pi(V)$ of a graph $G = (V, E)$ that minimizes $\sum_{u\in V} \max\left\{0, \max_{v\in N(u)} (\pi(u) - \pi(v))\right\}$. Let d_u be an integer variable used to measure the largest distance between u and its neighbor v with the lowest position in π that is also at a lower position than u (i.e., such that $\pi(v) < \pi(u)$). When $\pi(v) > \pi(u)$ for all $v \in N(u)$, we have $d_u = 0$. In other words, d_u measures the number of sets $\pi_{\leq,i}$ vertex u is a neighbor of. We obtain the following formulation.

Formulation 10 SumCut, Profile*: Minimize s subject to either Constraints* (1a) *and* (1b)*, or Constraints* (2a) *and* (2b)*, or Constraints* (3a) *and* (3b) *and*

$$\ell(u,v) \leq d_u \qquad \forall u \in V,\ \forall v \in N(u) \tag{10a}$$

$$0 \leq d_u \qquad \forall u \in V \tag{10b}$$

$$\sum_{u\in V} d_u \leq s \tag{10c}$$

where integer variable d_u measures the number of sets $\pi_{\leq,i}$ vertex u is a neighbor of, and $\ell(u,v)$ is either $\sum_{i=1}^{n} i \cdot \left(a_u^i - a_v^i\right)$, or $\sum_{i=1}^{n} \left(x_v^i - x_u^i\right)$, or $\sum_{w\in V\setminus\{u\}} \alpha_{w,u} - \sum_{w\in V\setminus\{v\}} \alpha_{w,v}$ when using respectively the variables of Formulations 1, 2 and 3.

With Constraints (10a) and (10b) we measure the value $d_u = \max\left\{0, \max_{v\in N(u)} (\pi(u) - \pi(v))\right\}$, that is the number of sets $\pi_{\leq,i}$ vertex u is a neighbor of, and with Constraint (10c) we count the sum of these memberships. The optimal solution of Formulation 10 is $\mathrm{sc}(G) = \mathrm{pr}(G)$. This formulation adds $O(n)$ integer variables and $O(n+m)$ constraints to the variables and constraints of Formulations 1 to 3.

In the following, we refer to the ILP formulations resulting from Formulation 10 when using the variables and constraints of respectively Formulations 1 to 3 as PR1, PR2, and PR3. The sizes of these ILP formulations are reported in Table 3.

4.5 Optimal linear arrangement

Given a graph $G = (V, E)$, the Optimal Linear Arrangement problem asks for a linear ordering $\pi \in \Pi(V)$ that minimizes the sum over all edges of the distance between the endpoints of the edges in the linear ordering π for G, and so the optimal value is $\mathrm{ola}(G) = \min_{L\in\Pi(V)} \sum_{uv\in E} \lambda(\pi, uv) = \min_{L\in\Pi(V)} \sum_{uv\in E} |\pi(u) - \pi(v)|$. The decision version of this optimization problem has been proved NP-complete in [15].

Several ILP formulations for the Optimal linear arrangement problem, based on Formulation 1, have already been proposed [1,2,6,29,34]. [2] presents two formulations, one using $O(n^6)$ variables and $O(n^5)$ constraints and the other using $O(n^4)$ variables and $O(n^3)$ constraints. The formulation presented in [6] uses $O(n^2+m)$ variables and $O(n^2m)$ constraints.

The following formulation is similar to the formulations presented in [29] and that is based on Formulation 1.

Formulation 11 Optimal Linear Arrangement*: Minimize o Subject to Constraints* (1a) *and* (1b)*, or Constraints* (2a) *and* (2b)*, or Constraints* (3a) *and* (3b) *and*

$$\ell(u,v) \leq d_{uv} \qquad \forall uv \in E \tag{11a}$$

$$\ell(u,v) \leq d_{uv} \qquad \forall uv \in E \tag{11b}$$

$$\sum_{uv\in E} d_{uv} \leq o \tag{11c}$$

where integer variable d_{uv} measure the distance between the endpoints of edge $uv \in E$, and $\ell(u,v)$ is either $\sum_{i=1}^{n} i \cdot \left(a_u^i - a_v^i\right)$, or $\sum_{i=1}^{n} \left(x_v^i - x_u^i\right)$, or $\sum_{w\in V\setminus\{u\}} \alpha_{w,u} - \sum_{w\in V\setminus\{v\}} \alpha_{w,v}$ when using respectively the variables of Formulations 1, 2 and 3.

With Constraints (11a) and (11b) we measure the distance d_{uv} between the endpoints of edge $uv \in E$, and with Constraint (11c) we count the sum of these distances. The optimal solution of Formulation 11 is $\mathrm{ola}(G)$. This formulation adds $O(m)$ integer variables and $O(m)$ constraints to the variables and constraints of Formulations 1 to 3.

In the following, we refer to the ILP formulations resulting from Formulation 11 when using the variables and constraints of respectively Formulations 1 to 3 as OLA1, OLA2, and OLA3. The sizes of these ILP formulations are reported in Table 3.

As reported in Table 1, we have $\sum_{uv\in E} \lambda(\pi, uv) = \sum_{i=1}^{n} \theta(\pi, i)$, and so the optimal value for the Optimal Linear Arrangement problem is also $\mathrm{ola}(G) = \min_{L\in\Pi(V)} \sum_{i=1}^{n} \theta(\pi, i)$ [12]. Therefore, we can derive another ILP formulation for this problem from the ILP formulations for the Cutwidth problem, replacing in Formulations 6 and 7 Constraint (6c) with

$\sum_{i=1}^{n} \sum_{uv \in E} z_{uv}^i \leq c$, and in Formulation 8 Constraint (8e) with $\sum_{i=1}^{n} c^i \leq c$. We refer to the ILP formulation resulting from the modification of Formulations 6 and 7 as respectively OLA'1, OLA'2, and OLA'3, and to the formulations resulting from the modification of Formulation 8 as OLA"1 and OLA"2.

5 Simulations and interpretation of results

In this section, we evaluate the performances of the ILP formulations presented in this paper. We start describing our experimental setting in Section 5.1. Then, we compare all proposed ILP formulations to previous proposals in Section 5.2 on small instances. Finally, in Section 5.3, we evaluate the best formulations for Bandwidth, Cutwidth and Pathwidth using a large data set of graphs.

5.1 Experimental setting

We have implemented all the ILP formulations presented in this paper, as well as formulations proposed by others, using the *SageMath* open-source mathematical software [36] combined with the CPLEX solver [21]. All computations have been performed on a computer equipped with two quad-cores Intel Xeon W5580 CPU operating at 3.20GHz and 64GB of RAM. We have forced the CPLEX solver to operate in sequential mode, setting parameter `CPXPARAM_Threads` to 1, and we have used the `@parallel` function decorator of *SageMath* to enable the simultaneous resolution of up to 8 instances.

In our experiments, we use two data sets. The first one, *Small* has already been used for instance in [20, 27, 28, 31] for evaluating ILP formulations and algorithms for Cutwidth. This data set consists of 84 graphs introduced in the context of the bandwidth reduction problem. The number of vertices ranges from 16 to 24, and the number of edges ranges from 18 to 49. This data set can be found as part of the CMPLIB [7]. The second one is the *Rome Graphs* data set [33], that has been introduced by the graph drawing research community. This data set has been used in [4] for the evaluation of ILP formulations for Bandwidth and Pathwidth. It consists of 11,534 undirected graphs of order $10 \leq n \leq 100$.

In all our experiments, the computation time is limited to 5 minutes (300 seconds). When a problem is not optimally solved within this time limit, we report the relative optimality gap (% gap), that is computed as $\frac{UB-LB}{UB}$, where UB is the value of the best found feasible solution (an upper bound) and LB is the best relaxed bound of the ILP (best lower bound for the problem). When the problem is optimally solved, this gap is set to 0.

The source code and the benchmark instances used to conduct our experiments can be found at `http://www-sop.inria.fr/members/David.Coudert/code/graph-linear-ordering/`.

5.2 Experiments with the *Small* data set

We summarize in Tables 4 to 8, respectively for Pathwidth, Cutwidth, Bandwidth, Sum-Cut and Optimal Linear Arrangement problems, the computational results of each formulations on the graphs of the *Small* data set. Detailed results per graphs can be found in Appendix A. For each problem and each formulation, we indicate the number of graphs of the data set for which the problem is optimally solved, the average of the optimality gaps over all graphs, and the average computation time. Observe that this average computation time is biased since we count 300 seconds for each unsolved instance, independently of its optimality gap (i.e., the same weight is given to an instance with a small optimality gap and to an instance with a large one).

	[4]	[13]	[35]	PW1	PW2	PW3
# opt	17	1	83	78	84	1
% gap	56.0	98.5	0.3	2.3	0.0	84.0
CPU time	258.5	297.5	33.6	56.0	1.7	299.9

Table 4: Pathwidth.

	[25]	[26]	CW1	CW2	CW3	CW'1	CW'2
# opt	1	3	56	84	0	66	68
% gap	94.5	80.3	9.5	0.0	91.8	6.7	8.0
CPU time	298.4	293.8	143.7	3.2	300.0	104.6	96.3

Table 5: Cutwidth.

	[26]	[4]	[31]	[31]+F2	[31]+F3	BW1	BW2	BW3
# opt	37	75	84	84	58	84	84	68
% gap	52.5	2.4	0.0	0.0	11.0	0.0	0.0	5.3
CPU time	198.9	99.5	1.3	1.9	153.1	13.6	2.2	108.2

Table 6: Bandwidth.

	SC1	SC2	SC3	PR1	PR2	PR3
# opt	17	65	0	5	35	9
% gap	37.0	1.7	99.6	50.2	13.8	43.5
CPU time	266.0	125.8	300.0	293.3	216.2	285.6

Table 7: SumCut.

	OLA1	OLA2	OLA3	OLA4	OLA5	OLA6	OLA7	OLA8	[31]	[31]+F2	[31]+F3
# opt	34	81	0	0	43	25	0	0	1	29	14
% gap	17.9	0.3	99.2	57.7	11.6	23.8	92.1	94.0	36.6	20.8	33.3
CPU time	218.8	51.7	300.0	300.0	195.7	250.3	300.0	300.0	298.9	234.1	271.4

Table 8: Optimal Linear Arrangement.

We can see from reported results that formulations based on Formulation 2, and more precisely PW2, CW2, BW2, SC2 and OLA2 are among the best formulations, if not the best. On the other hand, the formulations we have presented based on Formulation 3 are not efficient. Indeed, these formulations are able to solve very few instances compared to other proposals.

More specifically, we observe from Table 4 for the Pathwidth problem and from Table 5 the Cutwidth problem that not only PW2 and CW2 are the only formulations able to solve the problems on all the graphs of the *Small* data set, but also that reported average computation time are between one and two orders of magnitude lower than other formulations. This shows that Formulation 2 is much more suited for these two linear ordering optimization problems than other proposals.

The conclusion is different for the Bandwidth problem. In Table 6, formulations [31]+F2 and [31]+F3 are straightforward modifications of the formulation proposed in [31] to use the variables and constraints of respectively Formulations 2 and 3. Here, four formulations are able to solve the problem on all the graphs of the *Small* data set, and the fastest one is the formulation proposed in [31], relying on Formulation 1. In particular, we can observe the significant speedup of [31] over BW1 thanks to the introduction of integer variables recording the position of a vertex in the linear ordering. These variables offer fewer improvement for the formulations based on Formulation 2 (BW2 compared to [31]+F2), and are not helpful at all for the formulations based on Formulation 3. Although the formulation proposed in [31] has better performances than BW2 in this experiment, we will see in Section 5.3 that there is actually no clear winner between [31], [31]+F2 and BW2.

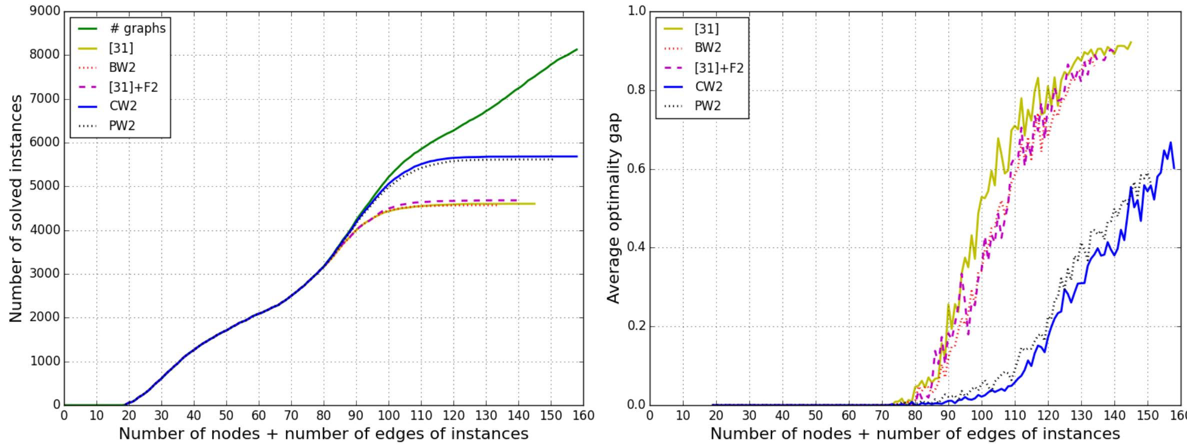
For the SumCut problem, formulation SC2 is not able to solve all instances within the specified time limit. However, it allows us to solve significantly more instances than using other formulations. Similarly, formulation OLA2 is the best available formulation for the Optimal Linear Arrangement problem, although some instances (3) were not solved. Note that in Table 8, formulations [31], [31]+F2, and [31]+F3 are straightforward adaptations of the formulations OLA1, OLA2 and OLA3 with the introduction of integer variables recording the position of a vertex in the linear ordering.

5.3 Experiments with the *Rome Graphs* data set

In this section we evaluate the best formulations for the Bandwidth, Cutwidth and Pathwidth problems on the *Rome Graphs* data set. More precisely, we report in Figure 1a the cumulative number of graphs for which optimal solutions are obtained using formulations PW2, CW2, BW2, [31] and [31]+F2, and in Figure 1b the average optimality gap of the unsolved instances. As in [4], we ordered the graphs by increasing size $n+m$ (number of nodes + number of edges) and ran computations starting from the smallest. Then, as soon as 400 consecutive instances in this order produced timeouts, we stop computations for the formulation and evaluated only the so far obtained results. This explains why the plots have different lengths. We also report in Table 9, for each of these models, the total number and the percentage of solved instances of the *Rome Graphs* data set (which contains 11,534 graphs), as well as the size of the last solved instance (which is also the largest solved).

	BW2	[31]	[31]+F2	CW2	PW2
# solved	4,568	4,603	4,681	5,683	5,615
% solved	39.6	39.9	40.6	49.3	48.7
Larger solved ($n+m$)	124=53+71	137=63+74	132=56+76	149=64+85	142=61+81

Table 9: Total number and percentage of solved instances of the *Rome Graphs* data set.



(a) Cumulative number of solved instances (b) Average optimality gap of unsolved instances

Figure 1: Cumulative number of solved instances and average optimality gap of unsolved instances of the Rome set of graphs for Bandwidth (Formulations [31], [31]+F2, and BW2), Cutwidth (CW2), and Pathwidth (PW2).

We observe from Table 9 and Figure 1a that the three models for the Bandwidth problem offer very similar performances on this data set with only 1% difference in the number of solved instances. As opposed to the results reported in Section 5.2 for the *Small* data set, here formulation [31]+F2 allows to solve slightly more instances than other models. Furthermore, we observe from Figure 1b that the average optimality gap of unsolved instances is lower for that formulation as well. This suggest that formulation [31]+F2 is asymptotically better than the other ones. In other words, longer computation times could lead to larger differences.

Concerning the Cutwidth and Pathwidth problems, we are able to solve almost all graphs such that $n + m < 90$, but very few with $n + m > 120$ within given time limit. This is a great improvement over [4] for Pathwidth. Indeed, the results reported in [4] for Pathwidth indicate that optimal solutions were obtained for only 17% of the instances of the *Rome Graphs* data set, solving almost all small graphs ($n + m < 45$) and almost no graphs with $n + m > 70$. Although such comparison lacks of fairness (different implementations, but comparable hardware), this confirm the results reported in Table 4 when using the *Small* data set.

6 Conclusion

In this paper we have presented different formulations for modeling linear ordering problems on graphs using integer linear programming. We have in particular proposed a new building block (Formulation 2) using variables expressing the membership of a vertex to a prefix rather than variables expressing the exact position of a vertex in the ordering (Formulation 1), or variables expressing a relative ordering (Formulation 3). Moreover, we have presented ILP formulations using less variables and contraints than other proposals. Then, we have compared these formulations using large benchmark instances. Our experiments show that Formulation 2 is very well suited for solving the Cutwidth and Pathwidth problems, and that it is also a good competitor for the Bandwidth problem. The performances of our formulations could certainly be improved with the addition of appropriate constraints (e.g., cutset inequalities).

We believe that Formulation 2 could help designing faster ILP formulations and branch-and-bound algorithms for solving various optimization problems in which a linear ordering of the vertices of a graph is required. In particular, it is straightforward to modify the ILP formulations presented in this paper for solving similar linear ordering optimization problems on directed graphs (e.g., Vertex Separation). Moreover, our work shall help improving the implementation of the `all_different` predicate in CP/SAT solvers.

References

[1] A. R. Amaral. On the exact solution of a facility layout problem. *European Journal of Operational Research*, 173(2):508 – 518, 2006.

[2] A. R. Amaral. A mixed 0-1 linear programming formulation for the exact solution of the minimum linear arrangement problem. *Optimization Letters*, 3(4):513–520, 2009.

[3] G. Appa, D. Magos, and I. Mourtos. LP relaxations of multiple all_different predicates. In J.-C. Régin and M. Rueher, editors, *Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems*, volume 3011 of *Lecture Notes in Computer Science*, pages 364–369. Springer Berlin Heidelberg, 2004.

[4] T. C. Biedl, T. Bläsius, B. Niedermann, M. Nöllenburg, R. Prutkin, and I. Rutter. Using ILP/SAT to determine pathwidth, visibility representations, and other grid-based graph drawings. In *Graph Drawing*, volume 8242 of *LNCS*, pages 460–471. Springer, 2013.

[5] H. L. Bodlaender, F. V. Fomin, A. M. Koster, D. Kratsch, and D. M. Thilikos. A note on exact algorithms for vertex ordering problems on graphs. *Theory Comput. Syst.*, 50(3):420–432, 2012.

[6] A. Caprara, A. N. Letchford, and J.-J. Salazar-González. Decorous lower bounds for minimum linear arrangement. *INFORMS Journal on Computing*, 23(1):26–40, 2011.

[7] CMPLIB. `http://www.optsicom.es/cutwidth/`.

[8] N. Cohen, D. Coudert, D. Mazauric, N. Nepomuceno, and N. Nisse. Tradeoffs in process strategy games with application in the WDM reconfiguration problem. *Theoretical Computer Science*, 412(35):4675–4687, 2011.

[9] D. Coudert, F. Huc, D. Mazauric, N. Nisse, and J.-S. Sereni. Reconfiguration of the routing in WDM networks with two classes of services. In *Optical Network Design and Modeling (ONDM)*, pages 1–6. IEEE, 2009.

[10] D. Coudert, S. Perennes, Q.-C. Pham, and J.-S. Sereni. Rerouting requests in WDM networks. In *Proc. AlgoTel*, pages 17–20, 2005.

[11] D. Coudert and J.-S. Sereni. Characterization of graphs and digraphs with small process number. *Discrete Applied Mathematics*, 159(11):1094–1109, 2011.

[12] J. Díaz, J. Petit, and M. Serna. A survey on graph layout problems. *ACM Computing Surveys*, 34(3):313–356, 2002.

[13] A. Duarte, L. F. Escudero, R. Martí, N. Mladenovic, J. J. Pantrigo, and J. Sánchez-Oro. Variable neighborhood search for the vertex separation problem. *Computers & Operations Research*, 39(12):3247–3255, 2012.

[14] M. Ganardi. Matching-based algorithms for computing treewidth. Master's thesis, Lehrune Forschungsgebeit Disktrete Optimierung, RWTH Aachen, 2012.

[15] M. Garey, D. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237 – 267, 1976.

[16] M. R. Garey, R. L. Graham, D. S. Johnson, and D. E. Knuth. Complexity results for bandwidth minimization. *SIAM Journal on Applied Mathematics*, 34(3):477–495, 1978.

[17] M. R. Garey and D. S. Johnson. *Computers and intractability.* W. H. Freeman and Co., San Francisco, Calif., 1979. A guide to the theory of NP-completeness, A Series of Books in the Mathematical Sciences.

[18] M. Grötschel, M. Jünger, and G. Reinelt. A cutting plane algorithm for the linear ordering problem. *Operations Research*, 32(6):1195–1220, 1984.

[19] M. Grötschel, M. Jünger, and G. Reinelt. Facets of the linear ordering polytope. *Mathematical Programming*, 33(1):43–60, 1985.

[20] F. Gurski. Linear programming formulations for computing graph layout parameters. *The Computer Journal*, 2015.

[21] IBM ILOG. CPLEX Optimization Studio 12.6.2. `http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/`.

[22] N. G. Kinnersley. The vertex separation number of a graph equals its pathwidth. *Information Processing Letters*, 42(6):345–350, 1992.

[23] A. M. C. A. Koster. ILP formulation for treewidth. private communication.

[24] J. Lee. All-different polytopes. *Journal of Combinatorial Optimization*, 6(3):335–352, 2002.

[25] M. C. López-Locés, N. Castillo-García, H. J. Fraire H., P. Bouvry, J. E. Pecero, R. A. Pazos Rangel, J. J. G. Barbosa, and F. Valdez. A new integer linear programming model for the cutwidth minimization problem of a connected undirected graph. In O. Castillo, P. Melin, W. Pedrycz, and J. Kacprzyk, editors, *Recent Advances on Hybrid Approaches for Designing Intelligent Systems*, volume 547 of *Studies in Computational Intelligence*, pages 509–517. Springer, 2014.

[26] J. Luttamaguzi, M. J. Pelsmajer, Z. Shen, and B. Yang. Integer programming methods for several optimization problems in graph theory. In G. Hu, editor, *20th International Conference on Computers and Their Applications (CATA)*, pages 50–55. ISCA, Mar. 2005.

[27] R. Martí, V. Campos, and E. Piñana. A branch and bound algorithm for the matrix bandwidth minimization. *European Journal of Operational Research*, 186(2):513 – 528, 2008.

[28] R. Martí, J. J. Pantrigo, A. Duarte, and E. G. Pardo. Branch and bound for the cutwidth minimization problem. *Computers & Operations Research*, 40(1):137 – 149, 2013.

[29] M. Moeini, S. Gueye, and S. Michel. A new mathematical model for the minimum linear arrangement problem. In B. Vitoriano, E. Pinson, and F. Valente, editors, *3rd International Conference on Operations Research and Enterprise Systems (ICORES)*, pages 57–62. SciTePress, Mar. 2014.

[30] J. Petit. Addenda to the survey of layout problems. *Bulletin of the European Association for Theoretical Computer Science*, (105):177–201, Oct. 2011.

[31] E. Piñana, I. Plana, V. Campos, and R. Martí. GRASP and path relinking for the matrix bandwidth minimization. *European Journal of Operational Research*, 153(1):200 – 210, 2004.

[32] J. Régin. A filtering algorithm for constraints of difference in CSPs. In B. Hayes-Roth and R. E. Korf, editors, *12th National Conference on Artificial Intelligence (AAAI)*, volume 1, pages 362–367. AAAI Press / The MIT Press, 1994.

[33] *Rome Graphs*. `http://www.graphdrawing.org/download/rome-graphml.tgz`.

[34] H. Seitz. *Contributions to the Minimum Linear Arrangement Problem*. PhD thesis, Heidelberg University, Apr. 2010.

[35] F. Solano and M. Pióro. Lightpath reconfiguration in WDM networks. *IEEE/OSA J. Opt. Commun. Netw.*, 2(12):1010–1021, 2010.

[36] W. Stein et al. *Sage Mathematics Software (Version 6.8)*. The Sage Development Team, 2015. `http://www.sagemath.org`.

[37] P. Surynek. Linear ordering in the SAT encoding of the all-different constraint over bit-vectors. *ALP Newsletter*, March/April, 2014.

[38] W. J. van Hoeve. The alldifferent constraint: A survey. *CoRR*, cs.PL/0105015, 2001.

[39] H. P. Williams and H. Yan. Representations of the all_different predicate of constraint satisfaction in integer programming. *INFORMS Journal on Computing*, 13(2):96–103, Mar. 2001.

A Detailed results for the *Small* data set

In this section, we report in Tables 10 to 14, respectively for Pathwidth, Cutwidth, Bandwidth, SumCut and Optimal Linear Arrangement problems the computational results of each presented formulations on the graphs of the *Small* data set. For each graph, we indicate its numbers of vertices and edges. For each graph and each formulation, we report the best found feasible solution, the optimality gap (%) and the computation time. When an instance is optimally solved, the measured width is highlighted in bold face. We use the symbol "–" to indicate that the computation time limit has been reached.

Table 10: Detailed results for PATHWIDTH on the *Small* data set.

			[4]			[13]			[35]			PW1			PW2			PW3		
Name	n	m	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time
p20_16_18	16	18	3	66.7	–	3	100.0	–	**3**	0.0	7.22	**3**	0.0	17.15	**3**	0.0	0.71	3	66.7	–
p19_16_19	16	19	3	66.7	–	3	86.7	–	**3**	0.0	3.59	**3**	0.0	5.64	**3**	0.0	0.33	3	58.2	–
p18_16_21	16	21	3	66.7	–	4	100.0	–	**3**	0.0	3.03	**3**	0.0	2.02	**3**	0.0	0.27	3	33.3	–
p17_16_24	16	24	4	75.0	–	5	100.0	–	**4**	0.0	2.89	**4**	0.0	2.85	**4**	0.0	0.47	5	74.0	–
p28_17_18	17	18	**2**	0.0	21.07	3	100.0	–	**2**	0.0	1.75	**2**	0.0	1.67	**2**	0.0	0.24	**2**	0.0	292.24
p29_17_18	17	18	**2**	0.0	6.84	**2**	0.0	86.05	**2**	0.0	1.78	**2**	0.0	1.25	**2**	0.0	0.25	2	50.0	–
p22_17_19	17	19	**2**	0.0	55.78	3	97.2	–	**2**	0.0	2.49	**2**	0.0	2.49	**2**	0.0	0.44	3	66.7	–
p26_17_19	17	19	**2**	0.0	163.93	3	100.0	–	**2**	0.0	2.01	**2**	0.0	2.57	**2**	0.0	0.35	4	75.0	–
p27_17_19	17	19	3	66.7	–	4	98.2	–	**3**	0.0	4.0	**3**	0.0	11.47	**3**	0.0	0.46	4	75.0	–
p30_17_19	17	19	3	66.7	–	4	100.0	–	**3**	0.0	7.29	**3**	0.0	10.47	**3**	0.0	0.56	4	75.0	–
p21_17_20	17	20	3	66.7	–	5	100.0	–	**3**	0.0	5.2	**3**	0.0	5.33	**3**	0.0	0.34	4	86.9	–
p25_17_20	17	20	3	66.7	–	4	100.0	–	**3**	0.0	3.11	**3**	0.0	4.39	**3**	0.0	0.48	4	75.0	–
p23_17_23	17	23	3	66.7	–	5	100.0	–	**3**	0.0	2.63	**3**	0.0	14.35	**3**	0.0	1.07	3	63.5	–
p24_17_29	17	29	4	75.0	–	6	98.6	–	**4**	0.0	3.43	**4**	0.0	8.78	**4**	0.0	1.78	5	79.7	–
p35_18_19	18	19	**2**	0.0	78.5	3	100.0	–	**2**	0.0	2.67	**2**	0.0	1.33	**2**	0.0	0.45	3	66.7	–
p38_18_19	18	19	**2**	0.0	15.98	4	100.0	–	**2**	0.0	1.46	**2**	0.0	1.73	**2**	0.0	0.31	3	66.7	–
p39_18_19	18	19	**2**	0.0	40.77	3	96.7	–	**2**	0.0	4.55	**2**	0.0	4.34	**2**	0.0	0.69	2	50.0	–
p32_18_20	18	20	3	66.7	–	3	100.0	–	**3**	0.0	12.53	**3**	0.0	13.28	**3**	0.0	0.65	4	74.4	–
p36_18_20	18	20	3	66.7	–	4	100.0	–	**3**	0.0	8.55	**3**	0.0	12.32	**3**	0.0	1.08	4	74.1	–
p37_18_20	18	20	3	66.7	–	3	100.0	–	**3**	0.0	7.77	**3**	0.0	11.19	**3**	0.0	0.75	3	66.7	–
p31_18_21	18	21	**2**	0.0	35.84	3	100.0	–	**2**	0.0	4.37	**2**	0.0	3.73	**2**	0.0	0.73	4	76.0	–
p33_18_21	18	21	3	66.7	–	4	100.0	–	**3**	0.0	4.54	**3**	0.0	3.3	**3**	0.0	0.57	4	73.5	–
p34_18_21	18	21	**2**	0.0	38.91	4	100.0	–	**2**	0.0	2.94	**2**	0.0	2.6	**2**	0.0	0.21	4	85.6	–
p40_18_32	18	32	5	80.0	–	7	100.0	–	**5**	0.0	11.43	**5**	0.0	20.46	**5**	0.0	1.32	6	84.2	–
p41_19_20	19	20	**2**	0.0	45.81	3	100.0	–	**2**	0.0	1.33	**2**	0.0	2.81	**2**	0.0	0.6	3	66.7	–
p46_19_20	19	20	**2**	0.0	97.43	4	100.0	–	**2**	0.0	3.58	**2**	0.0	3.5	**2**	0.0	0.75	3	66.7	–
p47_19_21	19	21	3	66.7	–	4	100.0	–	**3**	0.0	9.31	**3**	0.0	37.8	**3**	0.0	0.64	3	66.7	–
p48_19_21	19	21	**2**	0.0	163.23	3	100.0	–	**2**	0.0	5.16	**2**	0.0	4.41	**2**	0.0	0.46	4	79.8	–

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Table 10 – Continued from previous page

Name	n	m	[4] best	[4] gap	[4] time	[13] best	[13] gap	[13] time	[35] best	[35] gap	[35] time	PW1 best	PW1 gap	PW1 time	PW2 best	PW2 gap	PW2 time	PW3 best	PW3 gap	PW3 time
p43_19_22	19	22	3	66.7	–	5	100.0	–	**3**	0.0	10.52	**3**	0.0	8.38	**3**	0.0	0.82	4	96.0	–
p49_19_22	19	22	3	66.7	–	5	100.0	–	**3**	0.0	7.93	**3**	0.0	7.66	**3**	0.0	0.62	4	91.1	–
p42_19_24	19	24	4	75.0	–	5	100.0	–	**4**	0.0	19.96	**4**	0.0	46.35	**4**	0.0	1.26	4	100.0	–
p44_19_25	19	25	4	75.0	–	5	100.0	–	**4**	0.0	13.62	**4**	0.0	63.81	**4**	0.0	2.76	5	80.0	–
p45_19_25	19	25	3	66.7	–	5	100.0	–	**3**	0.0	11.97	**3**	0.0	10.63	**3**	0.0	0.7	4	77.0	–
p50_19_25	19	25	3	66.7	–	6	100.0	–	**3**	0.0	14.58	**3**	0.0	4.47	**3**	0.0	0.91	5	80.0	–
p58_20_21	20	21	3	66.7	–	4	100.0	–	**3**	0.0	9.59	**3**	0.0	38.64	**3**	0.0	1.39	3	66.7	–
p53_20_22	20	22	**2**	0.0	123.96	4	100.0	–	**2**	0.0	5.01	**2**	0.0	4.75	**2**	0.0	0.5	4	78.8	–
p60_20_22	20	22	3	66.7	–	4	100.0	–	**3**	0.0	12.54	**3**	0.0	27.67	**3**	0.0	1.13	4	75.0	–
p56_20_23	20	23	4	75.0	–	5	100.0	–	**4**	0.0	92.02	**4**	0.0	116.13	**4**	0.0	3.22	5	98.4	–
p59_20_23	20	23	3	66.7	–	4	100.0	–	**3**	0.0	17.75	**3**	0.0	51.96	**3**	0.0	1.24	4	96.2	–
p55_20_24	20	24	3	66.7	–	5	100.0	–	**3**	0.0	11.47	**3**	0.0	6.58	**3**	0.0	1.0	4	91.0	–
p57_20_24	20	24	3	66.7	–	5	100.0	–	**3**	0.0	6.21	**3**	0.0	9.5	**3**	0.0	0.68	4	96.6	–
p52_20_27	20	27	3	66.7	–	5	100.0	–	**3**	0.0	13.44	**3**	0.0	5.73	**3**	0.0	0.9	4	93.8	–
p51_20_28	20	28	5	80.0	–	7	100.0	–	**4**	0.0	13.34	**4**	0.0	32.34	**4**	0.0	0.98	6	98.3	–
p54_20_28	20	28	4	75.0	–	6	100.0	–	**3**	0.0	9.48	**3**	0.0	14.3	**3**	0.0	0.87	6	97.7	–
p61_21_22	21	22	3	66.7	–	4	100.0	–	**3**	0.0	15.53	**3**	0.0	103.98	**3**	0.0	1.53	3	83.3	–
p64_21_22	21	22	**2**	0.0	159.87	4	100.0	–	**2**	0.0	12.66	**2**	0.0	3.95	**2**	0.0	0.65	4	84.7	–
p67_21_22	21	22	**2**	0.0	198.95	4	98.4	–	**2**	0.0	7.74	**2**	0.0	3.08	**2**	0.0	0.95	5	94.2	–
p69_21_23	21	23	3	66.7	–	4	100.0	–	**3**	0.0	13.54	**3**	0.0	26.55	**3**	0.0	1.53	4	75.0	–
p65_21_24	21	24	3	66.7	–	5	100.0	–	**3**	0.0	29.26	**3**	0.0	17.5	**3**	0.0	1.17	5	80.0	–
p70_21_25	21	25	3	66.7	–	6	100.0	–	**3**	0.0	12.82	**3**	0.0	15.49	**3**	0.0	1.24	4	75.0	–
p68_21_27	21	27	3	66.7	–	7	100.0	–	**3**	0.0	17.19	**3**	0.0	30.17	**3**	0.0	1.02	5	100.0	–
p66_21_28	21	28	3	66.7	–	6	100.0	–	**3**	0.0	18.32	**3**	0.0	4.77	**3**	0.0	1.89	4	75.0	–
p62_21_30	21	30	4	75.0	–	6	100.0	–	**4**	0.0	22.99	**4**	0.0	45.47	**4**	0.0	1.63	7	85.7	–
p63_21_42	21	42	7	85.7	–	10	100.0	–	**6**	0.0	167.99	**6**	0.0	107.53	**6**	0.0	8.09	9	86.9	–
p75_22_25	22	25	3	66.7	–	5	100.0	–	**3**	0.0	29.29	**3**	0.0	63.23	**3**	0.0	0.98	5	94.2	–
p71_22_29	22	29	4	75.0	–	7	100.0	–	**4**	0.0	35.99	4	25.0	–	**4**	0.0	3.24	7	100.0	–
p73_22_29	22	29	4	75.0	–	7	100.0	–	**4**	0.0	176.3	4	60.0	–	**4**	0.0	11.23	6	100.0	–

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Table 10 – Continued from previous page

			[4]			[13]			[35]			PW1			PW2			PW3		
Name	n	m	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time
p79_22_29	22	29	3	66.7	–	8	100.0	–	**3**	0.0	13.2	**3**	0.0	10.69	**3**	0.0	1.4	7	89.6	–
p74_22_30	22	30	3	66.7	–	6	100.0	–	**3**	0.0	19.43	**3**	0.0	14.13	**3**	0.0	1.33	6	83.3	–
p76_22_30	22	30	3	66.7	–	7	98.8	–	**3**	0.0	38.68	**3**	0.0	11.12	**3**	0.0	1.6	5	98.0	–
p80_22_30	22	30	4	75.0	–	7	100.0	–	**4**	0.0	46.03	**4**	0.0	193.85	**4**	0.0	1.84	6	95.3	–
p78_22_31	22	31	4	75.0	–	7	100.0	–	**4**	0.0	134.88	4	25.0	–	**4**	0.0	2.65	7	100.0	–
p77_22_37	22	37	6	83.3	–	10	100.0	–	**5**	0.0	49.89	**5**	0.0	93.9	**5**	0.0	3.49	8	99.5	–
p72_22_49	22	49	8	87.5	–	15	100.0	–	**7**	0.0	191.36	**7**	0.0	211.51	**7**	0.0	4.0	9	98.2	–
p82_23_24	23	24	**2**	0.0	168.74	4	100.0	–	**2**	0.0	6.91	**2**	0.0	8.38	**2**	0.0	1.69	5	100.0	–
p83_23_24	23	24	**2**	0.0	196.09	5	100.0	–	**2**	0.0	7.39	**2**	0.0	10.37	**2**	0.0	1.35	5	87.7	–
p86_23_24	23	24	3	66.7	–	4	100.0	–	**2**	0.0	7.98	**2**	0.0	5.13	**2**	0.0	0.89	5	100.0	–
p84_23_26	23	26	3	66.7	–	5	100.0	–	**3**	0.0	104.02	**3**	0.0	74.11	**3**	0.0	1.92	4	97.1	–
p85_23_26	23	26	3	66.7	–	5	100.0	–	**3**	0.0	35.97	3	33.3	–	**3**	0.0	3.29	4	96.2	–
p88_23_26	23	26	3	66.7	–	5	100.0	–	**3**	0.0	30.22	**3**	0.0	185.66	**3**	0.0	1.15	6	100.0	–
p89_23_27	23	27	3	66.7	–	7	100.0	–	**3**	0.0	51.13	**3**	0.0	22.99	**3**	0.0	1.41	6	100.0	–
p87_23_30	23	30	4	75.0	–	22	100.0	–	**4**	0.0	47.85	**4**	0.0	176.68	**4**	0.0	2.2	7	95.4	–
p90_23_35	23	35	5	80.0	–	12	100.0	–	**4**	0.0	55.02	**4**	0.0	159.72	**4**	0.0	2.17	9	88.9	–
p81_23_46	23	46	8	87.5	–	22	100.0	–	**7**	0.0	201.25	7	22.4	–	**7**	0.0	14.32	11	100.0	–
p92_24_26	24	26	3	66.7	–	5	100.0	–	**3**	0.0	24.8	**3**	0.0	41.93	**3**	0.0	1.45	6	100.0	–
p97_24_26	24	26	2	50.0	–	7	100.0	–	**2**	0.0	24.96	**2**	0.0	11.16	**2**	0.0	2.08	5	100.0	–
p93_24_27	24	27	3	66.7	–	4	100.0	–	**2**	0.0	21.11	**2**	0.0	27.91	**2**	0.0	1.95	5	100.0	–
p95_24_27	24	27	3	66.7	–	6	100.0	–	**3**	0.0	31.52	**3**	0.0	55.62	**3**	0.0	1.57	5	99.5	–
p96_24_27	24	27	3	66.7	–	13	100.0	–	**3**	0.0	49.59	**3**	0.0	38.02	**3**	0.0	1.7	6	100.0	–
p99_24_27	24	27	3	66.7	–	6	100.0	–	**3**	0.0	37.47	**3**	0.0	21.65	**3**	0.0	1.44	4	100.0	–
p98_24_29	24	29	3	66.7	–	6	100.0	–	**3**	0.0	20.5	**3**	0.0	26.88	**3**	0.0	1.41	7	100.0	–
p94_24_31	24	31	4	75.0	–	5	100.0	–	**4**	0.0	263.08	4	25.0	–	**4**	0.0	2.9	7	100.0	–
p91_24_33	24	33	5	80.0	–	23	100.0	–	**4**	0.0	45.27	**4**	0.0	244.26	**4**	0.0	3.1	7	100.0	–
p100_24_34	24	34	5	80.0	–	9	100.0	–	4	25.0	–	**4**	0.0	173.36	**4**	0.0	3.79	6	100.0	–

Table 11: Detailed results for Cutwidth on the *Small* data set.

			[25]			[26]			CW1			CW2			CW3			CW'1			CW'2		
Name	n	m	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time
p20_16_18	16	18	4	50.0	–	**4**	0.0	40.63	**4**	0.0	13.97	**4**	0.0	0.42	6	84.7	–	**4**	0.0	3.67	**4**	0.0	4.48
p19_16_19	16	19	**4**	0.0	168.03	**4**	0.0	253.5	**4**	0.0	6.55	**4**	0.0	0.51	4	57.1	–	**4**	0.0	2.99	**4**	0.0	4.28
p18_16_21	16	21	5	46.7	–	5	60.0	–	**5**	0.0	19.11	**5**	0.0	0.53	7	70.8	–	**5**	0.0	10.01	**5**	0.0	7.98
p17_16_24	16	24	8	93.6	–	8	56.3	–	**7**	0.0	41.96	**7**	0.0	0.57	9	84.5	–	**7**	0.0	9.07	**7**	0.0	5.25
p28_17_18	17	18	4	75.0	–	4	50.0	–	**3**	0.0	1.87	**3**	0.0	0.39	4	79.1	–	**3**	0.0	2.39	**3**	0.0	1.83
p29_17_18	17	18	4	77.1	–	**3**	0.0	87.52	**3**	0.0	2.58	**3**	0.0	0.44	4	75.0	–	**3**	0.0	1.73	**3**	0.0	1.32
p22_17_19	17	19	4	87.4	–	4	37.5	–	**4**	0.0	23.93	**4**	0.0	1.35	5	80.0	–	**4**	0.0	3.99	**4**	0.0	7.29
p26_17_19	17	19	5	91.3	–	4	37.5	–	**4**	0.0	14.4	**4**	0.0	1.44	5	75.0	–	**4**	0.0	4.08	**4**	0.0	2.49
p27_17_19	17	19	5	91.9	–	4	25.0	–	**4**	0.0	21.85	**4**	0.0	0.42	5	89.7	–	**4**	0.0	6.87	**4**	0.0	4.37
p30_17_19	17	19	4	80.0	–	4	87.5	–	**4**	0.0	39.26	**4**	0.0	0.7	4	77.1	–	**4**	0.0	15.0	**4**	0.0	4.63
p21_17_20	17	20	5	77.3	–	5	80.0	–	**4**	0.0	9.85	**4**	0.0	0.63	5	77.9	–	**4**	0.0	3.97	**4**	0.0	2.83
p25_17_20	17	20	5	92.9	–	4	25.0	–	**4**	0.0	3.09	**4**	0.0	0.57	6	81.5	–	**4**	0.0	3.54	**4**	0.0	17.82
p23_17_23	17	23	7	97.1	–	6	50.0	–	**5**	0.0	18.08	**5**	0.0	0.7	6	66.7	–	**5**	0.0	3.77	**5**	0.0	5.26
p24_17_29	17	29	10	94.8	–	9	55.6	–	**8**	0.0	36.08	**8**	0.0	0.7	10	77.1	–	**8**	0.0	14.19	**8**	0.0	30.13
p35_18_19	18	19	5	95.0	–	4	25.0	–	**3**	0.0	9.22	**3**	0.0	0.78	5	90.0	–	**3**	0.0	2.36	**3**	0.0	5.11
p38_18_19	18	19	4	75.0	–	4	50.0	–	**3**	0.0	3.65	**3**	0.0	1.14	4	77.1	–	**3**	0.0	1.44	**3**	0.0	3.61
p39_18_19	18	19	5	86.7	–	4	25.0	–	**3**	0.0	14.19	**3**	0.0	0.58	4	75.0	–	**3**	0.0	2.23	**3**	0.0	18.82
p32_18_20	18	20	5	97.6	–	5	90.0	–	**4**	0.0	14.57	**4**	0.0	0.78	7	87.5	–	**4**	0.0	3.06	**4**	0.0	16.84
p36_18_20	18	20	5	90.0	–	4	87.5	–	**4**	0.0	11.43	**4**	0.0	0.59	5	80.0	–	**4**	0.0	15.99	**4**	0.0	3.51
p37_18_20	18	20	5	90.0	–	5	30.0	–	**4**	0.0	14.05	**4**	0.0	0.79	6	81.7	–	**4**	0.0	2.77	**4**	0.0	8.74
p31_18_21	18	21	4	87.9	–	5	50.0	–	**3**	0.0	13.42	**3**	0.0	0.45	6	86.8	–	**3**	0.0	2.28	**3**	0.0	3.29
p33_18_21	18	21	5	91.0	–	5	70.0	–	**4**	0.0	24.69	**4**	0.0	1.74	5	84.3	–	**4**	0.0	3.17	**4**	0.0	2.48
p34_18_21	18	21	5	88.5	–	5	60.0	–	**4**	0.0	12.74	**4**	0.0	0.58	7	91.7	–	**4**	0.0	10.51	**4**	0.0	7.59
p40_18_32	18	32	10	99.8	–	11	63.6	–	**8**	0.0	56.84	**8**	0.0	1.42	13	92.3	–	**8**	0.0	17.75	**8**	0.0	11.13
p41_19_20	19	20	5	97.1	–	4	25.0	–	**3**	0.0	5.17	**3**	0.0	0.62	6	91.1	–	**3**	0.0	3.93	**3**	0.0	2.8
p46_19_20	19	20	5	98.6	–	5	90.0	–	**3**	0.0	4.81	**3**	0.0	1.81	6	92.9	–	**3**	0.0	4.51	**3**	0.0	12.65
p47_19_21	19	21	4	90.4	–	5	100.0	–	**4**	0.0	192.49	**4**	0.0	3.84	5	87.5	–	**4**	0.0	101.92	**4**	0.0	27.0
p48_19_21	19	21	6	100.0	–	4	87.5	–	**4**	0.0	35.08	**4**	0.0	2.4	8	89.4	–	**4**	0.0	26.9	**4**	0.0	10.41

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Table 11 – Continued from previous page

Name	n	m	[25] best	[25] gap	[25] time	[26] best	[26] gap	[26] time	CW1 best	CW1 gap	CW1 time	CW2 best	CW2 gap	CW2 time	CW3 best	CW3 gap	CW3 time	CW'1 best	CW'1 gap	CW'1 time	CW'2 best	CW'2 gap	CW'2 time
p43_19_22	19	22	5	98.0	–	6	75.0	–	**4**	0.0	72.16	**4**	0.0	1.32	6	97.6	–	**4**	0.0	71.17	**4**	0.0	7.67
p49_19_22	19	22	7	100.0	–	5	40.0	–	**4**	0.0	8.77	**4**	0.0	0.83	6	95.0	–	**4**	0.0	22.31	**4**	0.0	32.09
p42_19_24	19	24	7	100.0	–	7	100.0	–	**5**	0.0	104.46	**5**	0.0	1.81	7	90.4	–	**5**	0.0	11.79	**5**	0.0	7.83
p44_19_25	19	25	7	100.0	–	7	100.0	–	**6**	0.0	40.21	**6**	0.0	0.72	8	87.3	–	**6**	0.0	29.74	**6**	0.0	22.27
p45_19_25	19	25	8	100.0	–	7	50.0	–	**5**	0.0	17.65	**5**	0.0	1.08	8	97.6	–	**5**	0.0	23.15	**5**	0.0	36.01
p50_19_25	19	25	6	93.4	–	7	78.6	–	**4**	0.0	47.07	**4**	0.0	1.6	6	94.0	–	**4**	0.0	4.93	**4**	0.0	2.93
p58_20_21	20	21	6	100.0	–	5	100.0	–	**3**	0.0	32.15	**3**	0.0	0.45	6	97.9	–	**3**	0.0	3.9	**3**	0.0	4.65
p53_20_22	20	22	7	100.0	–	5	50.0	–	**4**	0.0	186.63	**4**	0.0	1.17	6	98.0	–	**4**	0.0	16.7	**4**	0.0	20.32
p60_20_22	20	22	7	100.0	–	5	100.0	–	**4**	0.0	43.7	**4**	0.0	1.81	5	89.7	–	**4**	0.0	32.3	**4**	0.0	18.31
p56_20_23	20	23	6	100.0	–	6	100.0	–	5	20.0	–	**5**	0.0	2.66	7	95.7	–	5	20.0	–	**5**	0.0	67.69
p59_20_23	20	23	6	100.0	–	5	100.0	–	**4**	0.0	234.98	**4**	0.0	0.97	6	92.4	–	**4**	0.0	27.21	**4**	0.0	7.09
p55_20_24	20	24	6	100.0	–	6	100.0	–	**4**	0.0	46.38	**4**	0.0	0.91	6	84.9	–	**4**	0.0	5.91	**4**	0.0	20.44
p57_20_24	20	24	8	100.0	–	7	64.3	–	**4**	0.0	42.07	**4**	0.0	1.13	7	91.1	–	**4**	0.0	7.86	**4**	0.0	9.89
p52_20_27	20	27	8	100.0	–	7	100.0	–	**5**	0.0	16.72	**5**	0.0	0.79	9	88.9	–	**5**	0.0	5.53	**5**	0.0	13.44
p51_20_28	20	28	7	100.0	–	9	100.0	–	**6**	0.0	202.53	**6**	0.0	1.64	10	90.0	–	**6**	0.0	41.06	**6**	0.0	17.95
p54_20_28	20	28	7	100.0	–	8	100.0	–	6	16.7	–	**6**	0.0	2.72	10	100.0	–	**6**	0.0	169.14	**6**	0.0	269.02
p61_21_22	21	22	7	100.0	–	5	100.0	–	4	25.0	–	**4**	0.0	2.81	8	100.0	–	**4**	0.0	132.47	**4**	0.0	124.34
p64_21_22	21	22	8	100.0	–	6	100.0	–	**3**	0.0	109.24	**3**	0.0	0.92	7	96.9	–	**3**	0.0	6.93	**3**	0.0	9.42
p67_21_22	21	22	5	100.0	–	6	66.7	–	**3**	0.0	15.94	**3**	0.0	0.85	7	100.0	–	**3**	0.0	5.3	**3**	0.0	9.04
p69_21_23	21	23	7	100.0	–	6	100.0	–	5	20.0	–	**5**	0.0	4.78	8	100.0	–	5	20.0	–	5	40.0	–
p65_21_24	21	24	8	100.0	–	7	100.0	–	**4**	0.0	83.46	**4**	0.0	1.34	7	89.9	–	**4**	0.0	7.28	**4**	0.0	12.07
p70_21_25	21	25	8	100.0	–	7	100.0	–	5	20.0	–	**5**	0.0	2.67	9	100.0	–	**5**	0.0	96.9	**5**	0.0	162.74
p68_21_27	21	27	8	100.0	–	8	100.0	–	6	16.7	–	**6**	0.0	5.08	10	100.0	–	**6**	0.0	192.87	**6**	0.0	220.88
p66_21_28	21	28	9	100.0	–	10	100.0	–	6	16.7	–	**6**	0.0	6.27	8	87.5	–	6	16.7	–	6	16.7	–
p62_21_30	21	30	9	100.0	–	10	100.0	–	7	14.3	–	**7**	0.0	5.44	10	97.7	–	**7**	0.0	112.5	**7**	0.0	144.57
p63_21_42	21	42	17	100.0	–	15	100.0	–	12	30.3	–	**12**	0.0	4.96	18	100.0	–	12	41.7	–	12	66.7	–
p75_22_25	22	25	7	100.0	–	8	100.0	–	5	20.0	–	**5**	0.0	3.85	9	100.0	–	5	40.0	–	5	40.0	–
p71_22_29	22	29	9	100.0	–	7	100.0	–	5	40.0	–	**5**	0.0	6.66	8	100.0	–	**5**	0.0	27.73	**5**	0.0	73.61
p73_22_29	22	29	9	100.0	–	8	100.0	–	5	20.0	–	**5**	0.0	3.95	12	99.5	–	5	20.0	–	**5**	0.0	28.79

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Table 11 – Continued from previous page

Name	n	m	[25] best	[25] gap	[25] time	[26] best	[26] gap	[26] time	CW1 best	CW1 gap	CW1 time	CW2 best	CW2 gap	CW2 time	CW3 best	CW3 gap	CW3 time	CW'1 best	CW'1 gap	CW'1 time	CW'2 best	CW'2 gap	CW'2 time
p79_22_29	22	29	9	100.0	–	8	100.0	–	5	20.0	–	**5**	0.0	4.88	11	100.0	–	**5**	0.0	67.19	**5**	0.0	32.4
p74_22_30	22	30	10	100.0	–	9	100.0	–	6	16.7	–	**6**	0.0	4.3	11	100.0	–	6	16.7	–	6	33.3	–
p76_22_30	22	30	9	100.0	–	9	100.0	–	6	16.7	–	**6**	0.0	7.53	12	100.0	–	6	33.3	–	6	33.3	–
p80_22_30	22	30	8	100.0	–	11	100.0	–	**5**	0.0	254.29	**5**	0.0	4.63	10	97.5	–	**5**	0.0	84.27	**5**	0.0	26.44
p78_22_31	22	31	11	100.0	–	7	100.0	–	**6**	0.0	254.95	**6**	0.0	6.96	11	100.0	–	**6**	0.0	128.77	6	16.7	–
p77_22_37	22	37	14	100.0	–	12	100.0	–	8	25.0	–	**8**	0.0	2.46	15	100.0	–	**8**	0.0	115.57	**8**	0.0	66.5
p72_22_49	22	49	21	100.0	–	18	100.0	–	14	51.6	–	**14**	0.0	42.03	21	100.0	–	14	42.9	–	14	42.9	–
p82_23_24	23	24	8	100.0	–	5	100.0	–	**4**	0.0	29.96	**4**	0.0	1.06	10	96.7	–	**4**	0.0	132.47	**4**	0.0	121.47
p83_23_24	23	24	7	100.0	–	6	100.0	–	**4**	0.0	69.29	**4**	0.0	2.3	7	100.0	–	**4**	0.0	288.48	**4**	0.0	271.95
p86_23_24	23	24	6	100.0	–	6	100.0	–	**3**	0.0	20.52	**3**	0.0	2.82	7	100.0	–	**3**	0.0	10.41	**3**	0.0	22.1
p84_23_26	23	26	8	100.0	–	8	100.0	–	4	25.0	–	**4**	0.0	3.29	10	96.4	–	**4**	0.0	173.43	4	50.0	–
p85_23_26	23	26	9	100.0	–	7	100.0	–	**4**	0.0	262.3	**4**	0.0	1.81	9	100.0	–	**4**	0.0	181.97	**4**	0.0	38.11
p88_23_26	23	26	7	100.0	–	7	100.0	–	**4**	0.0	181.32	**4**	0.0	2.63	8	95.9	–	**4**	0.0	33.18	**4**	0.0	53.81
p89_23_27	23	27	9	100.0	–	8	100.0	–	5	20.0	–	**5**	0.0	5.38	8	87.5	–	**5**	0.0	205.47	**5**	0.0	68.9
p87_23_30	23	30	12	100.0	–	9	100.0	–	6	33.3	–	**6**	0.0	2.63	11	90.9	–	6	33.3	–	**6**	0.0	253.43
p90_23_35	23	35	11	100.0	–	11	100.0	–	7	42.9	–	**7**	0.0	4.29	14	100.0	–	7	28.6	–	7	14.3	–
p81_23_46	23	46	21	100.0	–	16	100.0	–	14	83.3	–	**13**	0.0	12.64	18	99.7	–	13	53.8	–	13	53.8	–
p92_24_26	24	26	8	100.0	–	7	100.0	–	4	50.0	–	**4**	0.0	2.96	10	100.0	–	**4**	0.0	70.45	**4**	0.0	70.65
p97_24_26	24	26	6	100.0	–	8	100.0	–	5	23.8	–	**4**	0.0	4.82	9	100.0	–	4	25.0	–	**4**	0.0	191.34
p93_24_27	24	27	7	100.0	–	8	100.0	–	**4**	0.0	71.95	**4**	0.0	4.85	9	100.0	–	**4**	0.0	118.0	4	25.0	–
p95_24_27	24	27	8	100.0	–	7	100.0	–	**4**	0.0	110.36	**4**	0.0	3.2	8	100.0	–	4	25.0	–	**4**	0.0	139.3
p96_24_27	24	27	7	100.0	–	9	100.0	–	4	25.0	–	**4**	0.0	2.32	13	100.0	–	**4**	0.0	273.03	**4**	0.0	170.3
p99_24_27	24	27	9	100.0	–	8	100.0	–	5	40.0	–	**5**	0.0	16.52	9	100.0	–	5	40.0	–	5	60.0	–
p98_24_29	24	29	10	100.0	–	9	100.0	–	**5**	0.0	241.65	**5**	0.0	5.83	11	100.0	–	**5**	0.0	161.69	5	40.0	–
p94_24_31	24	31	10	100.0	–	10	100.0	–	**6**	0.0	207.14	**6**	0.0	5.0	13	100.0	–	6	50.0	–	6	66.7	–
p91_24_33	24	33	13	100.0	–	8	100.0	–	6	35.0	–	**6**	0.0	6.84	10	100.0	–	6	16.7	–	**6**	0.0	182.05
p100_24_34	24	34	11	100.0	–	9	100.0	–	7	28.6	–	**7**	0.0	4.26	16	100.0	–	7	42.9	–	7	71.4	–

Table 12: Detailed results for BANDWIDTH on the *Small* data set.

			[26]			[4]			[31]			[31]+F2			[31]+F3			BW1			BW2			BW3		
Name	n	m	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time
p20_16_18	16	18	**4**	0.0	6.22	**4**	0.0	7.64	**4**	0.0	0.46	**4**	0.0	0.57	**4**	0.0	8.45	**4**	0.0	1.05	**4**	0.0	0.59	**4**	0.0	3.97
p19_16_19	16	19	**4**	0.0	7.19	**4**	0.0	10.26	**4**	0.0	0.48	**4**	0.0	0.63	**4**	0.0	13.6	**4**	0.0	1.34	**4**	0.0	0.55	**4**	0.0	3.37
p18_16_21	16	21	**4**	0.0	5.8	**4**	0.0	7.21	**4**	0.0	0.19	**4**	0.0	0.54	**4**	0.0	7.72	**4**	0.0	0.85	**4**	0.0	0.58	**4**	0.0	4.56
p17_16_24	16	24	**5**	0.0	13.92	**5**	0.0	25.14	**5**	0.0	0.56	**5**	0.0	0.74	**5**	0.0	15.29	**5**	0.0	2.19	**5**	0.0	0.69	**5**	0.0	18.42
p28_17_18	17	18	**3**	0.0	5.91	**3**	0.0	3.68	**3**	0.0	0.4	**3**	0.0	0.68	**3**	0.0	4.71	**3**	0.0	1.71	**3**	0.0	0.54	**3**	0.0	4.46
p29_17_18	17	18	**3**	0.0	8.89	**3**	0.0	6.89	**3**	0.0	0.29	**3**	0.0	0.52	**3**	0.0	7.59	**3**	0.0	0.99	**3**	0.0	0.47	**3**	0.0	4.2
p22_17_19	17	19	**4**	0.0	15.95	**4**	0.0	18.04	**4**	0.0	0.26	**4**	0.0	0.74	**4**	0.0	42.11	**4**	0.0	2.62	**4**	0.0	0.75	**4**	0.0	11.89
p26_17_19	17	19	**4**	0.0	13.92	**4**	0.0	15.78	**4**	0.0	0.33	**4**	0.0	0.64	**4**	0.0	29.21	**4**	0.0	1.9	**4**	0.0	0.63	**4**	0.0	13.83
p27_17_19	17	19	**4**	0.0	60.21	**4**	0.0	9.22	**4**	0.0	0.5	**4**	0.0	0.6	**4**	0.0	26.48	**4**	0.0	2.06	**4**	0.0	0.61	**4**	0.0	16.85
p30_17_19	17	19	**3**	0.0	8.01	**3**	0.0	10.94	**3**	0.0	0.39	**3**	0.0	0.51	**3**	0.0	6.17	**3**	0.0	1.45	**3**	0.0	0.62	**3**	0.0	5.57
p21_17_20	17	20	**4**	0.0	93.5	**4**	0.0	16.26	**4**	0.0	0.49	**4**	0.0	0.64	**4**	0.0	18.18	**4**	0.0	2.02	**4**	0.0	0.62	**4**	0.0	13.54
p25_17_20	17	20	**4**	0.0	9.66	**4**	0.0	10.59	**4**	0.0	0.68	**4**	0.0	0.66	**4**	0.0	11.79	**4**	0.0	1.4	**4**	0.0	0.65	**4**	0.0	14.82
p23_17_23	17	23	**4**	0.0	19.81	**4**	0.0	13.9	**4**	0.0	0.34	**4**	0.0	0.63	**4**	0.0	17.87	**4**	0.0	2.19	**4**	0.0	0.58	**4**	0.0	7.59
p24_17_29	17	29	6	83.3	–	**6**	0.0	38.88	**6**	0.0	1.46	**6**	0.0	1.22	**6**	0.0	177.55	**6**	0.0	7.0	**6**	0.0	1.65	**6**	0.0	91.69
p35_18_19	18	19	**4**	0.0	18.16	**4**	0.0	16.01	**4**	0.0	0.48	**4**	0.0	0.72	**4**	0.0	82.72	**4**	0.0	1.69	**4**	0.0	0.9	**4**	0.0	23.24
p38_18_19	18	19	**3**	0.0	7.8	**3**	0.0	12.57	**3**	0.0	0.36	**3**	0.0	0.58	**3**	0.0	18.36	**3**	0.0	2.11	**3**	0.0	0.49	**3**	0.0	5.24
p39_18_19	18	19	**3**	0.0	6.68	**3**	0.0	8.46	**3**	0.0	0.44	**3**	0.0	0.56	**3**	0.0	17.65	**3**	0.0	0.89	**3**	0.0	0.51	**3**	0.0	8.56
p32_18_20	18	20	**4**	0.0	11.45	**4**	0.0	13.91	**4**	0.0	0.54	**4**	0.0	0.65	**4**	0.0	21.11	**4**	0.0	1.82	**4**	0.0	0.82	**4**	0.0	12.67
p36_18_20	18	20	**4**	0.0	106.82	**4**	0.0	21.54	**4**	0.0	0.41	**4**	0.0	0.67	**4**	0.0	49.9	**4**	0.0	2.78	**4**	0.0	0.82	**4**	0.0	41.13
p37_18_20	18	20	**5**	0.0	253.29	**5**	0.0	34.19	**5**	0.0	1.11	**5**	0.0	1.44	**5**	0.0	224.21	**5**	0.0	5.86	**5**	0.0	1.03	**5**	0.0	107.21
p31_18_21	18	21	**4**	0.0	21.19	**4**	0.0	33.56	**4**	0.0	0.47	**4**	0.0	0.7	**4**	0.0	25.69	**4**	0.0	3.81	**4**	0.0	0.69	**4**	0.0	12.78
p33_18_21	18	21	**4**	0.0	114.09	**4**	0.0	27.17	**4**	0.0	0.34	**4**	0.0	0.7	**4**	0.0	17.55	**4**	0.0	1.31	**4**	0.0	0.72	**4**	0.0	27.54
p34_18_21	18	21	**4**	0.0	13.41	**4**	0.0	13.6	**4**	0.0	0.83	**4**	0.0	0.75	**4**	0.0	20.41	**4**	0.0	3.22	**4**	0.0	0.84	**4**	0.0	15.34
p40_18_32	18	32	**5**	0.0	24.65	**5**	0.0	31.1	**5**	0.0	0.9	**5**	0.0	1.23	**5**	0.0	33.98	**5**	0.0	4.54	**5**	0.0	0.7	**5**	0.0	14.92
p41_19_20	19	20	**3**	0.0	23.25	**3**	0.0	10.5	**3**	0.0	0.49	**3**	0.0	0.61	**3**	0.0	10.6	**3**	0.0	3.99	**3**	0.0	0.72	**3**	0.0	10.64
p46_19_20	19	20	**3**	0.0	107.31	**3**	0.0	19.96	**3**	0.0	0.51	**3**	0.0	0.64	**3**	0.0	9.55	**3**	0.0	4.63	**3**	0.0	0.72	**3**	0.0	6.57
p47_19_21	19	21	**3**	0.0	15.39	**3**	0.0	21.31	**3**	0.0	0.3	**3**	0.0	1.14	**3**	0.0	4.19	**3**	0.0	2.48	**3**	0.0	0.59	**3**	0.0	22.69
p48_19_21	19	21	**4**	0.0	127.29	**4**	0.0	22.16	**4**	0.0	0.48	**4**	0.0	0.85	**4**	0.0	45.61	**4**	0.0	3.25	**4**	0.0	0.78	**4**	0.0	25.65

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Table 12 – Continued from previous page

Name	n	m	[26]			[4]			[31]			[31]+F2			[31]+F3			BW1			BW2			BW3		
			best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time
p43_19_22	19	22	**4**	0.0	144.77	**4**	0.0	40.67	**4**	0.0	0.53	**4**	0.0	1.09	**4**	0.0	60.72	**4**	0.0	2.22	**4**	0.0	0.93	**4**	0.0	27.28
p49_19_22	19	22	**4**	0.0	181.73	**4**	0.0	20.25	**4**	0.0	0.48	**4**	0.0	1.01	**4**	0.0	21.82	**4**	0.0	2.63	**4**	0.0	0.74	**4**	0.0	19.04
p42_19_24	19	24	**4**	0.0	189.53	**4**	0.0	99.48	**4**	0.0	1.15	**4**	0.0	0.73	**4**	0.0	34.52	**4**	0.0	3.26	**4**	0.0	0.69	**4**	0.0	11.82
p44_19_25	19	25	5	100.0	–	**5**	0.0	136.25	**5**	0.0	1.14	**5**	0.0	1.57	**5**	0.0	169.95	**5**	0.0	5.47	**5**	0.0	1.23	**5**	0.0	43.0
p45_19_25	19	25	5	80.0	–	**5**	0.0	28.12	**5**	0.0	1.42	**5**	0.0	0.96	**5**	0.0	142.68	**5**	0.0	11.39	**5**	0.0	1.03	**5**	0.0	93.18
p50_19_25	19	25	**4**	0.0	27.78	**4**	0.0	32.49	**4**	0.0	1.29	**4**	0.0	0.72	**4**	0.0	29.48	**4**	0.0	3.55	**4**	0.0	0.97	**4**	0.0	18.82
p58_20_21	20	21	4	25.0	–	**4**	0.0	50.42	**4**	0.0	0.47	**4**	0.0	0.8	**4**	0.0	145.78	**4**	0.0	10.74	**4**	0.0	0.83	**4**	0.0	29.26
p53_20_22	20	22	**4**	0.0	230.73	**4**	0.0	24.68	**4**	0.0	0.69	**4**	0.0	0.72	**4**	0.0	64.7	**4**	0.0	1.69	**4**	0.0	0.92	**4**	0.0	40.48
p60_20_22	20	22	4	75.0	–	**4**	0.0	27.68	**4**	0.0	0.56	**4**	0.0	0.74	**4**	0.0	157.64	**4**	0.0	5.73	**4**	0.0	1.32	**4**	0.0	66.76
p56_20_23	20	23	4	25.0	–	**4**	0.0	37.96	**4**	0.0	0.64	**4**	0.0	0.91	**4**	0.0	44.0	**4**	0.0	12.06	**4**	0.0	0.68	**4**	0.0	25.5
p59_20_23	20	23	**4**	0.0	247.78	**4**	0.0	33.65	**4**	0.0	1.26	**4**	0.0	0.77	**4**	0.0	60.64	**4**	0.0	4.25	**4**	0.0	0.88	**4**	0.0	18.73
p55_20_24	20	24	**4**	0.0	129.89	**4**	0.0	49.6	**4**	0.0	0.49	**4**	0.0	0.61	**4**	0.0	90.89	**4**	0.0	6.63	**4**	0.0	1.77	**4**	0.0	80.93
p57_20_24	20	24	**4**	0.0	17.55	**4**	0.0	41.13	**4**	0.0	0.53	**4**	0.0	1.22	**4**	0.0	48.81	**4**	0.0	12.31	**4**	0.0	1.02	**4**	0.0	41.14
p52_20_27	20	27	5	95.0	–	**5**	0.0	23.41	**5**	0.0	1.41	**5**	0.0	2.99	**5**	0.0	100.57	**5**	0.0	6.24	**5**	0.0	3.07	**5**	0.0	141.42
p51_20_28	20	28	6	100.0	–	**5**	0.0	56.38	**5**	0.0	1.59	**5**	0.0	1.61	**5**	0.0	184.68	**5**	0.0	10.93	**5**	0.0	1.71	**5**	0.0	164.43
p54_20_28	20	28	5	94.9	–	**5**	0.0	74.16	**5**	0.0	1.55	**5**	0.0	1.1	**5**	0.0	286.71	**5**	0.0	10.51	**5**	0.0	1.13	**5**	0.0	60.92
p61_21_22	21	22	**4**	0.0	27.09	**4**	0.0	42.79	**4**	0.0	0.56	**4**	0.0	0.79	**4**	0.0	76.49	**4**	0.0	7.51	**4**	0.0	1.07	**4**	0.0	163.73
p64_21_22	21	22	4	93.2	–	**4**	0.0	43.32	**4**	0.0	0.54	**4**	0.0	0.83	**4**	0.0	110.19	**4**	0.0	11.12	**4**	0.0	1.48	**4**	0.0	56.92
p67_21_22	21	22	4	75.0	–	**4**	0.0	54.36	**4**	0.0	0.67	**4**	0.0	0.87	**4**	0.0	179.65	**4**	0.0	7.17	**4**	0.0	1.12	**4**	0.0	45.45
p69_21_23	21	23	4	94.2	–	**4**	0.0	43.86	**4**	0.0	0.57	**4**	0.0	1.24	5	20.0	–	**4**	0.0	21.94	**4**	0.0	1.67	**4**	0.0	59.09
p65_21_24	21	24	4	100.0	–	**4**	0.0	61.42	**4**	0.0	1.03	**4**	0.0	0.7	**4**	0.0	52.32	**4**	0.0	5.89	**4**	0.0	1.43	**4**	0.0	69.7
p70_21_25	21	25	**4**	0.0	288.27	**4**	0.0	44.08	**4**	0.0	1.07	**4**	0.0	2.54	**4**	0.0	54.71	**4**	0.0	8.66	**4**	0.0	1.42	**4**	0.0	22.27
p68_21_27	21	27	5	100.0	–	**5**	0.0	30.57	**5**	0.0	1.95	**5**	0.0	1.44	5	20.0	–	**5**	0.0	3.89	**5**	0.0	1.29	**5**	0.0	110.94
p66_21_28	21	28	5	100.0	–	**5**	0.0	173.52	**5**	0.0	0.97	**5**	0.0	2.16	5	20.0	–	**5**	0.0	12.48	**5**	0.0	2.05	**5**	0.0	104.49
p62_21_30	21	30	6	100.0	–	**5**	0.0	155.41	**5**	0.0	5.03	**5**	0.0	3.75	**5**	0.0	118.34	**5**	0.0	12.38	**5**	0.0	3.21	**5**	0.0	94.03
p63_21_42	21	42	11	100.0	–	7	14.3	–	**7**	0.0	1.13	**7**	0.0	2.76	7	38.1	–	**7**	0.0	53.2	**7**	0.0	8.08	7	14.3	–
p75_22_25	22	25	4	93.4	–	**4**	0.0	50.19	**4**	0.0	1.66	**4**	0.0	1.84	**4**	0.0	283.5	**4**	0.0	6.31	**4**	0.0	0.93	**4**	0.0	52.85
p71_22_29	22	29	5	100.0	–	**5**	0.0	263.73	**5**	0.0	3.13	**5**	0.0	2.02	**5**	0.0	212.13	**5**	0.0	30.07	**5**	0.0	2.86	5	20.0	–
p73_22_29	22	29	7	100.0	–	**5**	0.0	166.72	**5**	0.0	1.0	**5**	0.0	1.09	5	30.0	–	**5**	0.0	17.51	**5**	0.0	1.52	**5**	0.0	217.1

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Table 12 – Continued from previous page

Name	n	m	[26] best	[26] gap	[26] time	[4] best	[4] gap	[4] time	[31] best	[31] gap	[31] time	[31]+F2 best	[31]+F2 gap	[31]+F2 time	[31]+F3 best	[31]+F3 gap	[31]+F3 time	BW1 best	BW1 gap	BW1 time	BW2 best	BW2 gap	BW2 time	BW3 best	BW3 gap	BW3 time
p79_22_29	22	29	7	100.0	–	**4**	0.0	69.09	**4**	0.0	1.31	**4**	0.0	1.52	5	38.0	–	**4**	0.0	7.61	**4**	0.0	0.96	**4**	0.0	219.69
p74_22_30	22	30	5	100.0	–	**5**	0.0	41.19	**5**	0.0	2.24	**5**	0.0	1.63	5	20.0	–	**5**	0.0	37.04	**5**	0.0	1.31	**5**	0.0	211.92
p76_22_30	22	30	7	100.0	–	**5**	0.0	84.07	**5**	0.0	0.91	**5**	0.0	2.15	5	30.0	–	**5**	0.0	20.37	**5**	0.0	1.54	**5**	0.0	141.76
p80_22_30	22	30	5	100.0	–	**4**	0.0	83.55	**4**	0.0	1.74	**4**	0.0	1.31	5	40.0	–	**4**	0.0	11.98	**4**	0.0	1.97	**4**	0.0	135.47
p78_22_31	22	31	7	100.0	–	**5**	0.0	83.51	**5**	0.0	1.17	**5**	0.0	1.99	**5**	0.0	155.81	**5**	0.0	9.23	**5**	0.0	1.27	**5**	0.0	97.84
p77_22_37	22	37	8	100.0	–	6	16.7	–	**6**	0.0	1.68	**6**	0.0	2.77	6	33.6	–	**6**	0.0	86.78	**6**	0.0	2.55	6	28.4	–
p72_22_49	22	49	10	98.1	–	8	25.0	–	**8**	0.0	4.09	**8**	0.0	6.73	8	34.8	–	**8**	0.0	45.17	**8**	0.0	8.86	8	43.3	–
p82_23_24	23	24	5	96.5	–	**5**	0.0	51.28	**5**	0.0	0.84	**5**	0.0	1.62	5	30.0	–	**5**	0.0	9.14	**5**	0.0	2.53	5	23.3	–
p83_23_24	23	24	4	100.0	–	**4**	0.0	46.1	**4**	0.0	0.98	**4**	0.0	1.36	**4**	0.0	143.98	**4**	0.0	4.61	**4**	0.0	0.99	**4**	0.0	46.17
p86_23_24	23	24	4	100.0	–	**4**	0.0	73.04	**4**	0.0	1.71	**4**	0.0	1.87	4	25.0	–	**4**	0.0	9.16	**4**	0.0	1.34	**4**	0.0	111.28
p84_23_26	23	26	4	94.7	–	**4**	0.0	102.2	**4**	0.0	0.86	**4**	0.0	0.94	**4**	0.0	187.92	**4**	0.0	7.44	**4**	0.0	1.3	4	25.0	–
p85_23_26	23	26	5	100.0	–	**4**	0.0	39.88	**4**	0.0	1.69	**4**	0.0	1.45	**4**	0.0	288.62	**4**	0.0	10.04	**4**	0.0	1.22	**4**	0.0	226.94
p88_23_26	23	26	5	100.0	–	**4**	0.0	80.39	**4**	0.0	1.03	**4**	0.0	1.09	**4**	0.0	294.9	**4**	0.0	9.38	**4**	0.0	1.26	4	25.0	–
p89_23_27	23	27	7	100.0	–	5	20.0	–	**5**	0.0	1.83	**5**	0.0	2.42	5	21.3	–	**5**	0.0	21.5	**5**	0.0	1.79	5	20.0	–
p87_23_30	23	30	8	100.0	–	7	28.6	–	**6**	0.0	3.35	**6**	0.0	5.1	6	35.5	–	**6**	0.0	44.81	**6**	0.0	7.74	6	31.5	–
p90_23_35	23	35	10	100.0	–	5	20.0	–	**5**	0.0	2.18	**5**	0.0	2.37	6	34.9	–	**5**	0.0	23.61	**5**	0.0	3.51	**5**	0.0	137.02
p81_23_46	23	46	10	97.9	–	8	25.0	–	**8**	0.0	11.12	**8**	0.0	22.2	8	51.4	–	**8**	0.0	56.23	**8**	0.0	43.05	8	38.8	–
p92_24_26	24	26	5	96.1	–	**4**	0.0	126.68	**4**	0.0	1.28	**4**	0.0	2.17	4	37.3	–	**4**	0.0	13.07	**4**	0.0	1.54	**4**	0.0	127.98
p97_24_26	24	26	6	100.0	–	**4**	0.0	279.03	**4**	0.0	4.16	**4**	0.0	2.12	5	54.5	–	**4**	0.0	64.81	**4**	0.0	3.93	5	32.0	–
p93_24_27	24	27	6	100.0	–	**4**	0.0	67.86	**4**	0.0	1.66	**4**	0.0	1.09	**4**	0.0	291.29	**4**	0.0	7.27	**4**	0.0	1.91	**4**	0.0	100.79
p95_24_27	24	27	5	95.8	–	**5**	0.0	212.53	**5**	0.0	1.4	**5**	0.0	3.05	5	40.0	–	**5**	0.0	53.54	**5**	0.0	1.69	5	26.0	–
p96_24_27	24	27	4	100.0	–	**4**	0.0	220.26	**4**	0.0	0.93	**4**	0.0	2.58	4	25.0	–	**4**	0.0	10.71	**4**	0.0	3.14	**4**	0.0	227.05
p99_24_27	24	27	6	100.0	–	**4**	0.0	114.95	**4**	0.0	2.15	**4**	0.0	5.48	5	41.2	–	**4**	0.0	17.72	**4**	0.0	1.53	**4**	0.0	77.58
p98_24_29	24	29	5	100.0	–	**5**	0.0	114.38	**5**	0.0	1.5	**5**	0.0	3.02	5	40.0	–	**5**	0.0	14.17	**5**	0.0	2.42	5	28.0	–
p94_24_31	24	31	7	96.7	–	**5**	0.0	283.6	**5**	0.0	1.31	**5**	0.0	2.42	6	57.8	–	**5**	0.0	53.0	**5**	0.0	5.02	5	20.0	–
p91_24_33	24	33	7	100.0	–	8	37.5	–	**6**	0.0	2.99	**6**	0.0	8.29	6	55.8	–	**6**	0.0	60.19	**6**	0.0	9.58	6	36.7	–
p100_24_34	24	34	10	100.0	–	6	16.7	–	**6**	0.0	4.43	**6**	0.0	9.09	6	50.0	–	**6**	0.0	51.39	**6**	0.0	6.13	6	33.3	–

Table 13: Detailed results for SumCut on the *Small* data set.

Name	n	m	SC1 best	SC1 gap	SC1 time	SC2 best	SC2 gap	SC2 time	SC3 best	SC3 gap	SC3 time	PR1 best	PR1 gap	PR1 time	PR2 best	PR2 gap	PR2 time	PR3 best	PR3 gap	PR3 time
p20_16_18	16	18	**24**	0.0	104.92	**24**	0.0	8.38	25	96.9	–	**24**	0.0	163.01	**24**	0.0	31.65	**24**	0.0	141.27
p19_16_19	16	19	**27**	0.0	60.81	**27**	0.0	6.69	30	98.9	–	**27**	0.0	239.52	**27**	0.0	26.7	**27**	0.0	142.13
p18_16_21	16	21	**32**	0.0	209.33	**32**	0.0	18.46	34	99.8	–	32	28.8	–	**32**	0.0	122.91	32	27.0	–
p17_16_24	16	24	**41**	0.0	181.78	**41**	0.0	16.89	42	96.8	–	41	21.1	–	**41**	0.0	54.16	41	37.4	–
p28_17_18	17	18	**23**	0.0	97.19	**23**	0.0	10.89	28	100.0	–	23	24.7	–	**23**	0.0	40.24	**23**	0.0	213.7
p29_17_18	17	18	**20**	0.0	127.01	**20**	0.0	6.79	21	100.0	–	**20**	0.0	128.34	**20**	0.0	31.94	**20**	0.0	85.61
p22_17_19	17	19	**26**	0.0	46.76	**26**	0.0	1.83	30	96.0	–	**26**	0.0	169.16	**26**	0.0	21.11	**26**	0.0	107.41
p26_17_19	17	19	**27**	0.0	112.7	**27**	0.0	9.92	35	96.9	–	27	12.7	–	**27**	0.0	68.73	**27**	0.0	153.72
p27_17_19	17	19	**24**	0.0	34.76	**24**	0.0	3.09	24	99.5	–	**24**	0.0	237.61	**24**	0.0	16.47	**24**	0.0	167.54
p30_17_19	17	19	**29**	0.0	240.43	**29**	0.0	12.36	31	97.4	–	29	15.5	–	**29**	0.0	61.71	29	10.3	–
p21_17_20	17	20	**32**	0.0	204.44	**32**	0.0	13.69	32	99.7	–	32	24.4	–	**32**	0.0	34.48	32	30.3	–
p25_17_20	17	20	**27**	0.0	117.85	**27**	0.0	17.27	29	97.9	–	27	45.2	–	**27**	0.0	33.16	27	23.5	–
p23_17_23	17	23	31	9.0	–	**31**	0.0	24.16	33	100.0	–	31	19.8	–	**31**	0.0	62.45	31	14.5	–
p24_17_29	17	29	45	6.4	–	**45**	0.0	26.54	45	100.0	–	45	37.5	–	**45**	0.0	195.13	45	24.6	–
p35_18_19	18	19	27	10.4	–	**27**	0.0	22.21	27	99.0	–	27	29.6	–	**27**	0.0	59.55	27	14.0	–
p38_18_19	18	19	26	15.3	–	**26**	0.0	17.08	31	100.0	–	26	24.0	–	**26**	0.0	78.3	26	13.6	–
p39_18_19	18	19	**25**	0.0	262.74	**25**	0.0	33.24	26	98.2	–	25	34.7	–	**25**	0.0	63.94	25	19.3	–
p32_18_20	18	20	**29**	0.0	82.99	**29**	0.0	10.15	35	100.0	–	30	58.0	–	**29**	0.0	198.51	29	21.8	–
p36_18_20	18	20	33	48.5	–	**32**	0.0	31.25	33	100.0	–	33	43.2	–	**32**	0.0	169.94	32	23.3	–
p37_18_20	18	20	**30**	0.0	215.88	**30**	0.0	26.15	37	98.1	–	31	24.2	–	**30**	0.0	183.65	30	37.9	–
p31_18_21	18	21	**27**	0.0	61.62	**27**	0.0	3.9	29	100.0	–	27	41.1	–	**27**	0.0	22.5	**27**	0.0	268.99
p33_18_21	18	21	37	17.8	–	**37**	0.0	56.41	39	100.0	–	38	36.8	–	37	17.6	–	37	28.3	–
p34_18_21	18	21	**25**	0.0	81.64	**25**	0.0	3.36	25	100.0	–	25	35.8	–	**25**	0.0	22.91	**25**	0.0	211.79
p40_18_32	18	32	56	72.0	–	**55**	0.0	124.63	65	98.0	–	58	51.3	–	55	4.5	–	55	59.4	–
p41_19_20	19	20	26	45.0	–	**26**	0.0	269.9	32	100.0	–	26	26.9	–	26	9.6	–	26	37.1	–
p46_19_20	19	20	29	7.2	–	**29**	0.0	14.46	30	99.2	–	29	44.8	–	**29**	0.0	73.48	29	35.3	–
p47_19_21	19	21	30	11.3	–	**30**	0.0	16.22	42	99.1	–	30	50.0	–	**30**	0.0	84.98	30	23.0	–
p48_19_21	19	21	30	8.1	–	**30**	0.0	16.37	30	96.7	–	30	34.6	–	**30**	0.0	65.3	30	40.6	–

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Table 13 – Continued from previous page

Name	n	m	SC1 best	SC1 gap	SC1 time	SC2 best	SC2 gap	SC2 time	SC3 best	SC3 gap	SC3 time	PR1 best	PR1 gap	PR1 time	PR2 best	PR2 gap	PR2 time	PR3 best	PR3 gap	PR3 time
p43_19_22	19	22	32	14.3	–	**32**	0.0	20.01	35	100.0	–	32	45.7	–	**32**	0.0	77.66	32	40.0	–
p49_19_22	19	22	31	54.8	–	**31**	0.0	21.05	32	100.0	–	31	41.9	–	**31**	0.0	136.72	31	40.3	–
p42_19_24	19	24	47	65.5	–	**46**	0.0	134.75	59	100.0	–	46	62.4	–	46	4.9	–	46	38.4	–
p44_19_25	19	25	45	55.6	–	**45**	0.0	58.64	46	99.3	–	48	59.8	–	45	8.1	–	45	55.5	–
p45_19_25	19	25	35	17.6	–	**35**	0.0	75.2	36	99.8	–	36	55.7	–	35	5.7	–	36	37.7	–
p50_19_25	19	25	36	26.0	–	**36**	0.0	50.32	50	100.0	–	36	40.3	–	**36**	0.0	108.17	36	39.4	–
p58_20_21	20	21	30	20.6	–	**30**	0.0	41.0	43	100.0	–	31	35.5	–	**30**	0.0	129.2	30	44.3	–
p53_20_22	20	22	34	59.8	–	**32**	0.0	26.76	35	100.0	–	34	47.1	–	32	8.4	–	33	50.5	–
p60_20_22	20	22	34	22.9	–	**34**	0.0	75.2	37	100.0	–	34	26.1	–	34	7.8	–	34	35.0	–
p56_20_23	20	23	44	35.3	–	**43**	0.0	257.15	46	100.0	–	46	65.2	–	**43**	0.0	240.87	44	52.3	–
p59_20_23	20	23	34	26.0	–	**34**	0.0	192.32	53	100.0	–	35	45.0	–	34	10.1	–	34	51.5	–
p55_20_24	20	24	34	12.9	–	**34**	0.0	29.79	34	100.0	–	38	68.4	–	**34**	0.0	166.28	34	44.8	–
p57_20_24	20	24	35	51.4	–	**35**	0.0	21.54	36	100.0	–	35	45.7	–	**35**	0.0	145.01	35	40.5	–
p52_20_27	20	27	43	56.1	–	**42**	0.0	91.43	52	100.0	–	47	69.0	–	42	39.0	–	42	50.4	–
p51_20_28	20	28	56	73.2	–	54	3.4	–	59	100.0	–	54	63.7	–	54	15.4	–	54	51.4	–
p54_20_28	20	28	44	66.7	–	**43**	0.0	46.47	44	100.0	–	44	55.3	–	**43**	0.0	199.73	44	57.4	–
p61_21_22	21	22	32	21.1	–	**32**	0.0	171.65	36	100.0	–	33	59.1	–	32	15.6	–	32	44.0	–
p64_21_22	21	22	33	17.5	–	**33**	0.0	29.97	41	100.0	–	33	57.2	–	33	45.4	–	33	52.9	–
p67_21_22	21	22	31	27.7	–	**31**	0.0	39.74	56	100.0	–	32	63.3	–	**31**	0.0	198.29	32	48.5	–
p69_21_23	21	23	36	20.8	–	**36**	0.0	114.02	43	100.0	–	38	50.0	–	36	19.1	–	36	61.6	–
p65_21_24	21	24	39	63.4	–	**39**	0.0	63.46	45	100.0	–	39	55.9	–	39	29.3	–	39	37.5	–
p70_21_25	21	25	40	63.4	–	**40**	0.0	44.01	56	100.0	–	40	67.8	–	**40**	0.0	231.48	40	53.2	–
p68_21_27	21	27	46	32.7	–	**45**	0.0	92.86	55	100.0	–	50	68.3	–	45	15.1	–	47	64.5	–
p66_21_28	21	28	46	70.7	–	**45**	0.0	78.58	61	98.4	–	50	63.5	–	45	9.4	–	45	45.9	–
p62_21_30	21	30	55	34.4	–	54	1.8	–	54	100.0	–	54	63.3	–	54	16.4	–	54	55.0	–
p63_21_42	21	42	84	76.3	–	78	5.0	–	85	100.0	–	78	78.2	–	78	14.1	–	78	60.3	–
p75_22_25	22	25	38	24.8	–	**38**	0.0	128.52	54	100.0	–	38	61.2	–	38	10.9	–	38	57.3	–
p71_22_29	22	29	55	71.6	–	**53**	0.0	177.5	58	100.0	–	55	66.6	–	53	15.6	–	53	67.9	–
p73_22_29	22	29	49	75.1	–	48	6.1	–	65	100.0	–	52	74.3	–	48	21.9	–	48	61.3	–

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Table 13 – Continued from previous page

			SC1			SC2			SC3			PR1			PR2			PR3		
Name	n	m	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time
p79_22_29	22	29	53	46.0	–	50	2.6	–	64	100.0	–	51	68.6	–	50	16.8	–	50	54.0	–
p74_22_30	22	30	43	27.5	–	**43**	0.0	56.68	59	100.0	–	44	57.5	–	43	5.0	–	43	59.9	–
p76_22_30	22	30	45	66.7	–	45	20.8	–	57	100.0	–	48	71.6	–	45	23.3	–	45	66.7	–
p80_22_30	22	30	53	70.1	–	52	4.0	–	67	100.0	–	52	72.1	–	52	21.6	–	52	55.8	–
p78_22_31	22	31	50	31.7	–	**50**	0.0	102.94	62	100.0	–	55	70.9	–	50	8.5	–	50	59.6	–
p77_22_37	22	37	77	77.0	–	74	10.0	–	88	100.0	–	77	66.4	–	74	30.4	–	76	65.1	–
p72_22_49	22	49	104	83.7	–	101	13.6	–	110	100.0	–	102	70.9	–	101	34.2	–	103	82.7	–
p82_23_24	23	24	33	23.1	–	**33**	0.0	289.56	57	100.0	–	36	38.9	–	33	24.5	–	33	54.5	–
p83_23_24	23	24	32	33.6	–	**32**	0.0	173.52	53	100.0	–	34	60.6	–	32	27.6	–	34	62.4	–
p86_23_24	23	24	31	24.5	–	**31**	0.0	99.92	48	100.0	–	31	36.4	–	31	10.1	–	32	46.9	–
p84_23_26	23	26	39	69.2	–	**39**	0.0	260.88	57	100.0	–	40	73.3	–	39	50.3	–	40	52.9	–
p85_23_26	23	26	33	35.1	–	33	7.4	–	48	100.0	–	34	68.2	–	33	28.1	–	33	48.9	–
p88_23_26	23	26	44	71.8	–	43	5.5	–	58	100.0	–	49	67.2	–	43	30.3	–	45	63.1	–
p89_23_27	23	27	52	75.8	–	**50**	0.0	269.54	80	100.0	–	51	66.7	–	50	53.9	–	51	63.5	–
p87_23_30	23	30	60	74.6	–	59	8.1	–	88	100.0	–	61	75.7	–	61	39.8	–	60	66.7	–
p90_23_35	23	35	63	59.6	–	**63**	0.0	166.26	109	100.0	–	64	68.8	–	63	34.0	–	66	62.2	–
p81_23_46	23	46	111	79.9	–	102	17.1	–	110	100.0	–	103	73.0	–	102	44.7	–	105	83.1	–
p92_24_26	24	26	44	76.0	–	**42**	0.0	282.26	59	100.0	–	43	62.8	–	43	33.8	–	45	65.6	–
p97_24_26	24	26	37	65.6	–	**37**	0.0	89.11	65	100.0	–	44	65.5	–	37	22.2	–	37	61.2	–
p93_24_27	24	27	33	33.2	–	**33**	0.0	67.96	43	100.0	–	34	59.9	–	33	20.4	–	36	56.9	–
p95_24_27	24	27	41	43.8	–	41	4.9	–	69	100.0	–	42	71.2	–	41	26.1	–	41	62.7	–
p96_24_27	24	27	43	67.4	–	42	6.9	–	49	100.0	–	48	70.8	–	43	60.2	–	42	60.7	–
p99_24_27	24	27	45	75.8	–	43	8.3	–	80	100.0	–	45	71.3	–	43	27.3	–	43	63.3	–
p98_24_29	24	29	52	77.4	–	**45**	0.0	108.13	80	100.0	–	51	66.2	–	45	25.0	–	49	70.0	–
p94_24_31	24	31	59	56.8	–	56	7.0	–	100	100.0	–	62	75.8	–	57	56.8	–	56	67.9	–
p91_24_33	24	33	67	79.0	–	62	7.5	–	117	100.0	–	67	73.7	–	62	32.1	–	63	70.5	–
p100_24_34	24	34	60	73.3	–	57	5.7	–	87	100.0	–	57	73.3	–	57	26.4	–	57	67.8	–

Table 14: Detailed results for OPTIMAL LINEAR ARRANGEMENT on the *Small* data set.

			OLA1			OLA2			OLA3			OLA4			OLA5			OLA6			OLA7			OLA8			[31]			[31]+F2			[31]+F3		
Name	n	m	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time
p20_16_18	16	18	**36**	0.0	10.98	**36**	0.0	0.84	37	97.0	–	36	18.0	–	**36**	0.0	18.43	**36**	0.0	34.18	40	83.9	–	38	79.0	–	**36**	0.0	203.78	**36**	0.0	34.29	**36**	0.0	29.85
p19_16_19	16	19	**39**	0.0	52.2	**39**	0.0	6.27	40	95.3	–	41	40.6	–	**39**	0.0	20.63	**39**	0.0	71.29	39	78.5	–	44	87.8	–	39	13.1	–	**39**	0.0	57.74	**39**	0.0	51.1
p18_16_21	16	21	**48**	0.0	24.64	**48**	0.0	2.7	49	92.3	–	48	39.6	–	**48**	0.0	58.5	**48**	0.0	34.77	51	84.3	–	48	85.4	–	48	8.5	–	**48**	0.0	54.91	**48**	0.0	69.94
p17_16_24	16	24	**70**	0.0	216.3	**70**	0.0	11.31	72	95.5	–	71	52.3	–	**70**	0.0	53.78	**70**	0.0	184.84	72	83.3	–	70	91.7	–	71	23.5	–	**70**	0.0	167.65	70	10.9	–
p28_17_18	17	18	**32**	0.0	15.78	**32**	0.0	1.7	37	97.3	–	32	23.4	–	**32**	0.0	15.35	**32**	0.0	31.38	38	80.4	–	32	85.4	–	32	9.9	–	**32**	0.0	27.7	**32**	0.0	99.55
p29_17_18	17	18	**29**	0.0	96.86	**29**	0.0	7.44	30	96.7	–	29	24.3	–	**29**	0.0	45.37	**29**	0.0	115.28	29	73.0	–	29	79.3	–	29	17.4	–	**29**	0.0	109.18	**29**	0.0	121.78
p22_17_19	17	19	**37**	0.0	31.02	**37**	0.0	3.11	38	100.0	–	37	27.0	–	**37**	0.0	27.46	**37**	0.0	98.95	40	77.5	–	42	88.1	–	37	17.7	–	**37**	0.0	33.35	**37**	0.0	93.18
p26_17_19	17	19	**40**	0.0	23.4	**40**	0.0	2.27	44	98.3	–	40	57.5	–	**40**	0.0	39.51	**40**	0.0	99.07	44	88.6	–	44	92.2	–	40	21.3	–	**40**	0.0	76.64	**40**	0.0	139.47
p27_17_19	17	19	**39**	0.0	33.49	**39**	0.0	6.29	52	97.8	–	39	45.7	–	**39**	0.0	29.58	**39**	0.0	68.22	39	91.6	–	39	91.0	–	39	9.9	–	**39**	0.0	54.58	**39**	0.0	195.72
p30_17_19	17	19	**39**	0.0	88.28	**39**	0.0	8.09	46	93.0	–	39	40.8	–	**39**	0.0	38.72	**39**	0.0	115.81	41	84.0	–	39	85.9	–	39	21.1	–	**39**	0.0	157.96	39	11.6	–
p21_17_20	17	20	**44**	0.0	43.33	**44**	0.0	4.34	49	96.9	–	44	45.8	–	**44**	0.0	23.79	**44**	0.0	129.13	46	84.8	–	44	91.2	–	44	18.8	–	**44**	0.0	54.33	**44**	0.0	105.68
p25_17_20	17	20	**40**	0.0	30.14	**40**	0.0	2.89	42	96.7	–	40	39.8	–	**40**	0.0	14.76	**40**	0.0	32.72	45	90.6	–	46	89.1	–	40	14.4	–	**40**	0.0	33.2	40	6.2	–
p23_17_23	17	23	**51**	0.0	64.74	**51**	0.0	7.78	51	99.0	–	51	34.5	–	**51**	0.0	32.35	**51**	0.0	140.55	51	81.5	–	51	88.9	–	51	15.9	–	**51**	0.0	66.22	51	15.8	–
p24_17_29	17	29	**82**	0.0	60.44	**82**	0.0	12.63	85	97.3	–	84	55.9	–	**82**	0.0	45.31	**82**	0.0	96.67	82	87.8	–	82	92.7	–	82	24.1	–	**82**	0.0	103.1	82	27.6	–
p35_18_19	18	19	**35**	0.0	263.22	**35**	0.0	3.41	42	99.5	–	36	34.4	–	**35**	0.0	49.97	**35**	0.0	114.37	38	88.8	–	38	89.5	–	35	12.6	–	**35**	0.0	117.2	**35**	0.0	254.68
p38_18_19	18	19	**32**	0.0	104.26	**32**	0.0	9.74	32	97.0	–	32	17.5	–	**32**	0.0	84.05	**32**	0.0	176.08	32	75.0	–	32	82.3	–	32	25.6	–	**32**	0.0	57.8	32	7.4	–
p39_18_19	18	19	**36**	0.0	65.08	**36**	0.0	10.72	42	99.2	–	36	38.0	–	**36**	0.0	63.19	**36**	0.0	143.47	36	86.1	–	43	91.1	–	36	20.9	–	**36**	0.0	162.03	**36**	0.0	169.1
p32_18_20	18	20	**46**	0.0	81.77	**46**	0.0	13.0	49	99.1	–	47	51.3	–	**46**	0.0	176.02	**46**	0.0	295.49	47	91.5	–	53	92.4	–	47	38.4	–	46	15.2	–	46	24.7	–
p36_18_20	18	20	**44**	0.0	129.04	**44**	0.0	9.23	47	98.5	–	44	31.9	–	**44**	0.0	107.31	44	4.0	–	48	88.5	–	47	89.4	–	44	24.0	–	**44**	0.0	194.32	46	35.9	–
p37_18_20	18	20	**48**	0.0	62.85	**48**	0.0	2.39	61	98.4	–	49	51.2	–	**48**	0.0	201.99	**48**	0.0	269.52	51	90.2	–	51	96.1	–	48	32.4	–	48	17.0	–	48	17.3	–
p31_18_21	18	21	**39**	0.0	79.08	**39**	0.0	2.19	47	100.0	–	39	52.2	–	**39**	0.0	24.69	**39**	0.0	234.74	43	83.7	–	45	91.1	–	39	15.9	–	**39**	0.0	64.81	**39**	0.0	96.91
p33_18_21	18	21	**47**	0.0	213.88	**47**	0.0	12.13	62	99.5	–	49	40.8	–	**47**	0.0	63.9	47	7.9	–	51	88.7	–	53	90.4	–	47	24.3	–	**47**	0.0	145.38	47	10.2	–
p34_18_21	18	21	**39**	0.0	30.73	**39**	0.0	3.51	45	97.2	–	39	40.5	–	**39**	0.0	95.38	**39**	0.0	181.25	52	91.3	–	39	88.9	–	39	18.0	–	**39**	0.0	34.31	**39**	0.0	147.64
p40_18_32	18	32	93	14.4	–	**93**	0.0	26.25	108	99.8	–	93	56.8	–	**93**	0.0	104.24	93	11.6	–	96	90.6	–	93	94.6	–	93	27.8	–	93	7.3	–	93	6.5	–
p41_19_20	19	20	36	9.6	–	**36**	0.0	42.09	41	99.9	–	36	36.1	–	**36**	0.0	117.2	36	13.9	–	51	95.0	–	37	87.4	–	36	39.6	–	36	16.1	–	36	35.9	–
p46_19_20	19	20	**37**	0.0	48.33	**37**	0.0	2.27	43	99.8	–	37	29.8	–	**37**	0.0	53.52	**37**	0.0	154.43	39	90.2	–	47	95.5	–	37	22.3	–	**37**	0.0	161.03	37	12.6	–
p47_19_21	19	21	39	14.7	–	**39**	0.0	14.7	44	100.0	–	39	65.7	–	**39**	0.0	78.78	39	12.3	–	42	90.5	–	39	89.7	–	39	19.3	–	**39**	0.0	69.11	39	11.8	–
p48_19_21	19	21	**41**	0.0	221.17	**41**	0.0	10.36	45	97.7	–	41	55.9	–	**41**	0.0	210.93	**41**	0.0	207.64	46	90.3	–	44	86.4	–	41	27.2	–	**41**	0.0	179.43	41	11.0	–
p43_19_22	19	22	**42**	0.0	196.56	**42**	0.0	4.86	46	99.8	–	42	54.5	–	**42**	0.0	55.29	42	8.1	–	48	88.1	–	51	94.1	–	42	26.5	–	42	2.4	–	42	29.2	–
p49_19_22	19	22	**45**	0.0	68.42	**45**	0.0	2.56	54	99.9	–	45	55.6	–	**45**	0.0	80.94	**45**	0.0	118.77	45	88.9	–	45	91.0	–	45	26.0	–	**45**	0.0	151.27	45	10.5	–
p42_19_24	19	24	59	51.1	–	**57**	0.0	31.66	72	98.7	–	59	51.0	–	**57**	0.0	106.81	57	7.7	–	61	90.2	–	57	90.5	–	57	26.4	–	**57**	0.0	192.87	57	38.1	–
p44_19_25	19	25	70	13.3	–	**70**	0.0	74.78	79	98.4	–	70	60.8	–	70	5.1	–	70	11.4	–	75	90.9	–	71	93.8	–	72	41.3	–	70	27.0	–	70	37.4	–
p45_19_25	19	25	**56**	0.0	35.55	**56**	0.0	3.21	81	97.8	–	59	64.9	–	**56**	0.0	45.27	**56**	0.0	176.79	69	94.1	–	70	94.0	–	56	24.1	–	56	2.7	–	**56**	0.0	227.1
p50_19_25	19	25	51	3.5	–	**51**	0.0	24.84	54	100.0	–	51	29.4	–	51	11.8	–	51	4.6	–	66	91.7	–	56	92.3	–	51	20.8	–	**51**	0.0	131.45	51	10.0	–
p58_20_21	20	21	**41**	0.0	189.65	**41**	0.0	15.49	50	100.0	–	41	58.5	–	**41**	0.0	83.29	41	24.7	–	47	91.5	–	52	97.3	–	43	34.5	–	**41**	0.0	202.16	44	41.0	–
p53_20_22	20	22	47	14.3	–	**47**	0.0	46.01	48	100.0	–	48	56.5	–	47	6.0	–	47	18.8	–	59	94.6	–	48	91.7	–	47	39.0	–	47	24.1	–	47	20.2	–
p60_20_22	20	22	47	8.7	–	**47**	0.0	16.11	59	100.0	–	48	33.2	–	47	4.6	–	48	30.2	–	49	88.4	–	49	94.4	–	48	41.6	–	47	8.5	–	47	54.6	–
p56_20_23	20	23	57	40.1	–	**54**	0.0	47.64	55	100.0	–	59	63.3	–	54	16.9	–	54	41.2	–	65	90.8	–	60	94.3	–	55	42.1	–	55	43.6	–	58	44.7	–
p59_20_23	20	23	**50**	0.0	217.87	**50**	0.0	20.79	77	100.0	–	51	58.4	–	50	8.0	–	50	34.0	–	58	96.5	–	52	90.4	–	50	36.8	–	50	33.1	–	52	40.8	–
p55_20_24	20	24	**51**	0.0	75.92	**51**	0.0	13.57	74	99.4	–	51	59.7	–	**51**	0.0	91.48	51	3.9	–	78	97.4	–	57	91.2	–	51	30.0	–	**51**	0.0	266.94	51	51.7	–
p57_20_24	20	24	**56**	0.0	98.09	**56**	0.0	12.03	56	100.0	–	56	55.6	–	**56**	0.0	129.12	56	21.0	–	66	92.9	–	56	94.5	–	56	35.4	–	56	23.5	–	56	27.9	–
p52_20_27	20	27	71	12.1	–	**70**	0.0	19.34	84	99.6	–	74	67.9	–	**70**	0.0	260.86	70	14.4	–	73	95.4	–	71	97.0	–	70	36.9	–	70	43.0	–	70	24.6	–
p51_20_28	20	28	73	15.3	–	**73**	0.0	38.18	95	100.0	–	74	59.0	–	**73**	0.0	139.77	74	15.5	–	88	95.5	–	78	96.2	–	75	33.3	–	73	22.9	–	74	27.9	–
p54_20_28	20	28	68	4.0	–	**68**	0.0	35.81	80	100.0	–	68	61.8	–	**68**	0.0	136.04	68	6.9	–	84	93.5	–	85	97.7	–	68	31.2	–	68	28.6	–	68	20.3	–
p61_21_22	21	22	44	15.4	–	**44**	0.0	33.17	50	100.0	–	44	53.2	–	44	9.6	–	44	27.3	–	54	94.4	–	49	98.0	–	44	46.3	–	44	30.1	–	44	58.5	–
p64_21_22	21	22	45	8.4	–	**45**	0.0	17.87	58	100.0	–	47	68.8	–	45	3.3	–	45	35.2	–	55	94.5	–	54	98.2	–	47	47.6	–	45	22.1	–	45	42.2	–

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Table 14 – Continued from previous page

Name	n	m	OLA1			OLA2			OLA3			OLA4			OLA5			OLA6			OLA7			OLA8			[31]			[31]+F2			[31]+F3		
			best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time	best	gap	time
p67_21_22	21	22	**40**	0.0	125.3	**40**	0.0	14.64	52	100.0	–	41	70.6	–	**40**	0.0	133.02	40	18.5	–	40	92.5	–	46	93.4	–	41	35.4	–	40	41.2	–	40	41.3	–
p69_21_23	21	23	53	8.2	–	**53**	0.0	32.88	64	100.0	–	54	65.1	–	53	23.1	–	53	35.9	–	67	94.0	–	65	98.2	–	54	49.9	–	53	21.0	–	53	40.1	–
p65_21_24	21	24	52	54.1	–	**51**	0.0	58.39	61	100.0	–	52	68.5	–	**51**	0.0	296.44	51	33.0	–	55	92.7	–	67	97.3	–	52	37.4	–	51	11.8	–	51	30.9	–
p70_21_25	21	25	57	12.9	–	**57**	0.0	29.49	66	100.0	–	58	75.2	–	**57**	0.0	220.27	57	36.6	–	61	93.9	–	58	98.2	–	57	44.1	–	57	33.8	–	57	22.4	–
p68_21_27	21	27	68	8.4	–	**68**	0.0	14.49	81	99.2	–	72	63.2	–	68	1.5	–	68	23.8	–	83	95.0	–	68	95.9	–	68	46.3	–	68	5.9	–	70	42.1	–
p66_21_28	21	28	77	26.0	–	**77**	0.0	66.75	99	100.0	–	78	64.4	–	77	19.4	–	77	44.4	–	86	95.3	–	77	95.2	–	77	55.9	–	77	20.1	–	77	45.1	–
p62_21_30	21	30	91	38.2	–	**91**	0.0	45.17	114	99.9	–	91	69.7	–	91	39.8	–	91	40.7	–	117	96.6	–	106	98.1	–	93	52.9	–	91	30.9	–	91	55.3	–
p63_21_42	21	42	150	41.2	–	**148**	0.0	217.5	205	100.0	–	151	73.6	–	148	26.2	–	148	47.0	–	173	97.1	–	170	97.7	–	153	48.3	–	152	26.4	–	161	51.9	–
p75_22_25	22	25	56	19.4	–	**55**	0.0	60.27	77	100.0	–	56	78.0	–	56	18.7	–	56	31.3	–	64	95.3	–	59	96.5	–	56	41.6	–	55	8.6	–	55	28.2	–
p71_22_29	22	29	69	45.2	–	**69**	0.0	43.7	101	100.0	–	86	69.4	–	**69**	0.0	278.35	69	32.8	–	91	96.7	–	82	96.3	–	69	35.3	–	69	32.5	–	69	39.3	–
p73_22_29	22	29	67	23.4	–	**65**	0.0	28.82	86	100.0	–	65	65.7	–	65	6.5	–	65	43.3	–	76	97.4	–	77	97.4	–	65	36.4	–	65	39.7	–	65	26.2	–
p79_22_29	22	29	67	28.0	–	**67**	0.0	60.52	129	100.0	–	68	61.2	–	67	11.4	–	67	44.0	–	79	94.9	–	67	97.0	–	67	48.2	–	67	38.1	–	67	55.6	–
p74_22_30	22	30	**67**	0.0	284.17	**67**	0.0	9.54	103	100.0	–	68	75.0	–	**67**	0.0	183.72	67	13.9	–	98	95.9	–	71	98.6	–	67	35.2	–	67	29.7	–	67	25.4	–
p76_22_30	22	30	74	30.7	–	**73**	0.0	93.69	114	100.0	–	75	68.5	–	73	15.1	–	73	51.7	–	81	95.1	–	76	98.0	–	76	44.1	–	73	8.4	–	73	40.0	–
p80_22_30	22	30	75	57.3	–	**74**	0.0	50.17	122	100.0	–	74	66.5	–	74	12.5	–	74	46.3	–	98	95.9	–	92	97.0	–	74	38.9	–	74	43.2	–	74	50.7	–
p78_22_31	22	31	79	17.2	–	**79**	0.0	65.54	112	100.0	–	82	69.4	–	79	10.8	–	79	35.1	–	105	97.1	–	90	98.9	–	79	42.3	–	79	44.4	–	79	47.5	–
p77_22_37	22	37	122	42.0	–	**122**	0.0	160.27	166	100.0	–	125	74.2	–	122	49.9	–	122	62.7	–	144	97.2	–	135	97.8	–	124	55.2	–	128	48.5	–	127	69.7	–
p72_22_49	22	49	215	83.2	–	195	9.7	–	262	99.9	–	198	81.7	–	195	67.1	–	195	55.4	–	223	97.3	–	230	98.1	–	208	47.7	–	202	55.9	–	195	53.8	–
p82_23_24	23	24	56	17.9	–	**56**	0.0	57.39	82	100.0	–	56	65.7	–	56	17.6	–	56	48.2	–	70	95.7	–	65	98.5	–	58	61.0	–	56	23.2	–	56	50.0	–
p83_23_24	23	24	49	12.0	–	**48**	0.0	33.66	89	100.0	–	48	67.4	–	48	10.0	–	50	33.0	–	76	97.4	–	60	98.3	–	54	60.6	–	48	48.1	–	49	59.5	–
p86_23_24	23	24	43	8.1	–	**43**	0.0	23.66	79	100.0	–	43	76.4	–	43	13.1	–	43	24.2	–	60	95.7	–	50	98.0	–	44	49.3	–	43	12.3	–	44	43.2	–
p84_23_26	23	26	53	34.4	–	**52**	0.0	223.62	86	100.0	–	55	69.7	–	52	15.7	–	52	44.0	–	62	95.2	–	54	96.3	–	52	52.3	–	53	39.6	–	52	50.3	–
p85_23_26	23	26	50	24.6	–	**50**	0.0	27.48	90	100.0	–	50	68.0	–	50	26.3	–	50	46.5	–	77	96.1	–	70	96.0	–	54	52.8	–	50	55.4	–	50	51.0	–
p88_23_26	23	26	59	24.7	–	**58**	0.0	66.52	103	100.0	–	59	59.3	–	60	20.0	–	60	46.7	–	69	94.2	–	71	97.1	–	60	54.6	–	59	19.1	–	60	53.3	–
p89_23_27	23	27	71	47.9	–	**71**	0.0	142.82	122	100.0	–	72	67.5	–	71	50.7	–	71	57.8	–	91	96.7	–	83	97.6	–	72	60.5	–	71	23.9	–	72	68.3	–
p87_23_30	23	30	86	51.4	–	**83**	0.0	77.96	123	100.0	–	87	71.0	–	89	54.8	–	83	51.2	–	103	97.1	–	102	98.0	–	85	60.0	–	85	55.6	–	84	53.6	–
p90_23_35	23	35	105	59.1	–	**100**	0.0	105.37	158	100.0	–	100	65.7	–	100	24.9	–	100	54.0	–	136	97.1	–	102	96.1	–	100	51.4	–	100	42.1	–	100	59.3	–
p81_23_46	23	46	185	77.8	–	179	13.0	–	293	100.0	–	189	76.2	–	184	37.9	–	179	55.9	–	199	97.5	–	196	98.5	–	181	53.9	–	179	37.1	–	191	61.8	–
p92_24_26	24	26	58	38.2	–	**58**	0.0	203.88	96	100.0	–	61	72.8	–	58	32.1	–	58	54.3	–	66	95.5	–	69	98.6	–	60	58.7	–	58	58.7	–	59	64.4	–
p97_24_26	24	26	58	21.1	–	**57**	0.0	86.21	123	100.0	–	59	69.6	–	58	27.2	–	59	52.0	–	73	95.9	–	60	97.9	–	59	52.2	–	58	63.9	–	58	63.8	–
p93_24_27	24	27	55	5.0	–	**55**	0.0	30.55	81	100.0	–	61	78.4	–	55	19.0	–	55	34.5	–	90	97.8	–	61	98.2	–	55	43.0	–	55	40.0	–	57	63.2	–
p95_24_27	24	27	60	53.3	–	**59**	0.0	79.72	110	100.0	–	59	81.8	–	59	39.8	–	59	39.8	–	90	97.8	–	82	97.6	–	62	49.9	–	59	20.6	–	62	65.4	–
p96_24_27	24	27	53	20.2	–	**53**	0.0	178.39	73	100.0	–	60	70.2	–	58	60.8	–	53	35.9	–	67	97.0	–	54	96.3	–	55	54.9	–	53	51.9	–	58	66.0	–
p99_24_27	24	27	65	50.4	–	**62**	0.0	80.5	98	100.0	–	65	80.4	–	64	58.0	–	63	49.2	–	99	97.0	–	62	96.8	–	62	51.7	–	62	19.6	–	75	69.6	–
p98_24_29	24	29	74	32.4	–	**71**	0.0	96.43	107	100.0	–	79	72.0	–	71	22.1	–	75	52.0	–	106	98.1	–	74	98.7	–	73	63.3	–	71	55.3	–	71	65.7	–
p94_24_31	24	31	87	62.3	–	**84**	0.0	83.49	119	100.0	–	84	70.0	–	84	19.2	–	89	59.3	–	114	97.0	–	91	98.7	–	90	55.2	–	90	52.4	–	86	70.4	–
p91_24_33	24	33	101	35.9	–	96	3.3	–	136	100.0	–	101	69.6	–	97	34.3	–	99	55.4	–	131	97.7	–	120	99.2	–	96	48.1	–	105	64.2	–	99	70.2	–
p100_24_34	24	34	98	64.2	–	**98**	0.0	97.67	157	100.0	–	101	75.2	–	98	26.4	–	98	51.5	–	118	96.6	–	125	98.4	–	104	60.7	–	98	58.5	–	99	61.4	–